Integration :- If $x(t) \xleftrightarrow{FS} a_k$

and $a_0 = 0$, then

$$y(t) = \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{FS} b_k = \frac{1}{jk\omega_0} a_k$$

Proof: Let $x(t)$ be a periodic signal. we have the Fourier series Representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Integrating

$$y(t) = \int_{-\infty}^{t} x(t)\,dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} a_k e^{jk\omega_0 t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{a_k e^{jk\omega_0 t}}{jk\omega_0} = \sum_{k=-\infty}^{\infty} \left(\frac{a_k}{jk\omega_0}\right) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$\therefore b_k = \frac{1}{jk\omega_0} a_k$$

So if $\boxed{y(t) = \int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{FS} b_k = \frac{1}{jk\omega_0} a_k}$

Differentiation $x(t) \xleftrightarrow{FS} a_k$

then $\boxed{y(t) = \frac{dx(t)}{dt} \xleftrightarrow{FS} b_k = jk\omega_0 a_k}$

Proof $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k (jk\omega_0) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \quad \text{where } b_k = (jk\omega_0) a_k$$

So if $y(t) = \frac{dx(t)}{dt} \xleftrightarrow{FS} \boxed{b_k = (jk\omega_0) a_k}$

Periodic Convolution

$$x(t) \xleftrightarrow{FS} a_k \quad , \quad y(t) \xleftrightarrow{FS} b_k$$

then $z(t) = x(t) * y(t) = \frac{1}{T}\int_0^T x(\tau)\, y(t-\tau)\, d\tau \longleftrightarrow z_k = a_k b_k$

Proof :- The Periodic convolution is defined by

$$z(t) = \frac{1}{T}\int_0^T x(\tau)\, y(t-\tau)\, d\tau$$

In terms of Fourier Series coefficients

$$z_k = \frac{1}{T}\int_0^T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T \left(\frac{1}{T}\int_0^T x(\tau)\, y(t-\tau)\, d\tau \right) e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T x(\tau) \left(\frac{1}{T}\int_0^T y(t-\tau)\, e^{-jk\omega_0 t}\, dt \right) d\tau.$$

From time shifting Property

if $y(t) \xleftrightarrow{FS} b_k$, then $y(t-\tau) \xleftrightarrow{FS} b_k e^{-jk\omega_0 \tau}$

We have

$$\cancel{z_k = \frac{1}{T}\int_0^T x(\tau) e^{-jk\omega_0 \tau}\, d\tau\, b_k} \quad \left\{ \because \cancel{b_k = \frac{1}{T}\int_0^T y(t)\, e^{-j\omega_0 t}\, dt} \right.$$

$$\therefore \int_0^T y(t-\tau)\, e^{-jk\omega_0 t}\, dt = b_k\, e^{-jk\omega_0 \tau}$$

$$\therefore z_k = \underbrace{\frac{1}{T}\int_0^T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau}_{a_k}\, b_k = a_k b_k.$$

β

Parseval's Relation for Continuous Time Periodic Signals

If $x(t) \longleftrightarrow a_k$.

then $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = a_0^2 + 2\sum_{k=1}^{\infty} |a_k|^2$

It states that the total av. power in a periodic signals equals the sum of av. powers in all of its harmonic components.

Proof

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_0^T x(t)\, x^*(t)\, dt$$

$$= \frac{1}{T}\int_0^T x(t)\left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right)^* dt$$

$$= \frac{1}{T}\int_0^T x(t)\left(\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\right) dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \int_0^T x(t) e^{-jk\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k^* a_k$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2 = a_0^2 + \sum_{k=1}^{\infty} 2|a_k|^2$$

Properties Discrete time Fourier Series

Property	Periodic Signal	Fourier Series coefficients
	$x[n]$,	a_k
	$y[n]$	b_k.
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Freq. shifting	$e^{jM(\frac{2\pi}{N})n} x[n]$	a_{k-M}.
Conjugation	$x^*[n]$	a^*_{-k}
Time reversal	$x[-n]$	a_{-k}.
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$	$\frac{1}{m} a_k$ (Viewed as periodic with

7) Periodic Convolution $\sum_{r=\langle N\rangle} x[r]\,y[n-r]$ ~~$a_k b_k$~~ $N a_k b_k$

1) Multiplication $x[n]\,y[n]$ $\sum_{l=\langle N\rangle} a_l\, b_{k-l}$

First Diff. $x[n]-x[n-1]$ $(1-e^{-jk(2\pi/N)})a_k$

Running Sum $\sum_{k=-\infty}^{\infty} x[k]$ $\left(\frac{1}{1-e^{-jk(2\pi/N)}}\right)a_k$

Conjugate symmetry for real signals $x[n]$ real

$$\begin{cases} a_k = a_{-k}^* \\ Re\{a_k\} = Re[a_{-k}\} \\ Img\{a_k\} = -Im\{a_{-k}\} \\ |a_k| = |a_{-k}| \\ \measuredangle a_k = -\measuredangle a_{-k}\} \end{cases}$$

Real and Even Signals $x[n]$ real & even a_k real & even

Real and Odd Signals $x[n]$ real and odd a_k purely imaginary and odd

Even & odd Decomposition of Real Signals $\begin{cases} x_e[n] = Ev\{x[n]\} \\ x_o[n] = Od\{x[n]\} \end{cases}$ $Re\{a_k\}$ $j\,Im[a_k]$

Parseval's Relation $= \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$