Fourier Integral :- So far we have expanded periodic functions in series of sines, cosines and complex exponent. Physically, we could think of the terms of these Fourier series as representing a set of harmonics. In music there would be an infinite set of frequencies $n\omega$, $n=1,2,3$, notice that this set, although infinite, does not by any means include all possible frequencies. In electricity, a Fourier series could represent a periodic voltage; again we could think of this as made up of an infi but discrete set of a-c voltages of frequencies $n\omega$. Similarly, in discussing light, a Fourier series could represent light consisting of a discrete set of wavelength λ/n, $n=1,2,\ldots$, that is, a discrete set of colours. Two relate questions occurs. First, is it possible to represent a func which is not periodic by something analogous to a Fourier series? Second, can we somehow extend a mod Fourier series to cover the case of a continuous spectrum of wavelengths of light, or a sound wave containing a continuous set of frequencies?

If we recall, an integral is a limit of a sum, Therefore in all the above cases, the Fourier series (that is, a sum of terms) is replaced by a Fourier integral. The Fourier integral can be used to represent nonperiodic functions, for example a single voltage pulse not repeated, or a flash of light, or a sound which is not repeated. The Fourier integral also represents a continuous set (spectrum) of frequencies.

The fourier series of periodic function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--- (1)}$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} f(t)\,dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}\,dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{n\pi t}{l}\,dt$$

Substituting values of Fourier coefficients a_0, a_n and b_n in (1)

$$f(x) = \frac{1}{2l}\int_{-l} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{l}\int_{-l} f(t)\cos\frac{n\pi t}{l}\,dt\ \cos\frac{n\pi x}{l}$$

$$+\sum_{n=1}^{\infty}\frac{1}{l}\int_{-l}^{+l} f(t)\sin\frac{n\pi t}{l}\,dt\ \sin\frac{n\pi x}{l}$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\,dt + \frac{1}{l}\int_{-l}^{l} f(t)\left[\sum_{n=1}^{\infty}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2l}\int_{-l}^{l} f(t)\left[1 + 2\sum \cos\frac{n\pi(x-t)}{l}\right]dt$$

$$\text{or } f(x) = \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\frac{\pi}{l} + \sum_{n=1}^{\infty}\frac{2\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{+l} f(t)\left[\frac{\pi}{l} + \sum_{n=1}^{\infty}\frac{\pi}{l}\cos\left(\frac{n\pi(x-t)}{l}\right) + \sum_{n=1}^{\infty}\frac{\pi}{l}\cos\left(\frac{-n\pi(x-t)}{l}\right)\right.$$

since $2\cos\theta = \cos\theta + (\cos -\theta)$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\sum_{n=1}^{\infty}\frac{\pi}{l}\cos\frac{-n\pi(x-t)}{l} + \frac{\pi}{l}\cos\left(\frac{0\cdot\pi(x-t)}{l}\right)\right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{+l} f(t)\left[\sum_{n=-\infty}^{+\infty}\frac{\pi}{l}\cos\frac{n\pi(x-t)}{l}\right]dt$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(t)\left[\sum_{n=-\infty}^{+\infty}\frac{1}{l/\pi}\cos\left\{\frac{n}{l/\pi}(x-t)\right\}\right]dt \quad —(2)$$

When l becomes infinitely large i.e. as $l\to\infty$, $\frac{l}{\pi}\to\infty$, we have

$$\lim_{l\to\infty}\sum_{n=-\infty}^{+\infty}\frac{1}{l/\pi}\cos\left\{\frac{n}{l/\pi}(x-t)\right\}$$

$$= \lim_{\Delta u\to 0}\sum_{n=-\infty}^{+\infty}\Delta u\cos\{n\,\Delta u(x-t)\} \quad \text{where } \Delta u = \frac{1}{l/\pi}$$

$$= \int_{-\infty}^{+\infty}\cos\{u(x-t)\}\,du \quad —(3)$$

(by the definition of integral as the limit of a sum

Using (3), Equation (2) become.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\left[\int_{-\infty}^{+\infty}\cos u(x-t)\,du\right]dt \quad —(4)$$

This double integral is known as Fourier integral and holds if x is a point of continuity of $f(x)$.

The second integral in (4) may be written as

$$\int_{-\infty}^{+\infty}\cos u(x-t)\,du = \int_{-\infty}^{0}\cos u(x-t)\,du + \int_{0}^{\infty}\cos u(x-t)\,du$$

$$= -\int_{0}^{-\infty}\cos u(x-t)\,du + \int_{0}^{\infty}\cos u(x-t)\,du$$

If we replace u by $-u$ in the first term, we get

$$\int_{-\infty}^{+\infty}\cos u(x-t)\,du = 2\int_{0}^{\infty}\cos u(x-t)\,du \quad \text{— (5)}$$

Therefore Fourier integral (4) may also be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\left[2\int_{0}^{\infty}\cos u(x-t)\,du\right]dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\,dt \int_{0}^{\infty}\cos u(x-t)\,du \quad \text{— (6)}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt \quad \text{— (7)}$$

Equation (7) represents another form of Fourier Integral

~~Remark.- As cos x is an even function of x, eqn (6) and (7) may be expressed as~~

$$\cancel{f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}}$$

Now let us consider the terms of Fourier integral for odd, even and complex functions.

Case (1) :- If $f(x)$ is an odd function of x i.e. $f(-x) = -f(x)$, o

$$\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt = \cancel{\int\!\!\int f(t)\cos}\int_{-\infty}^{0} f(t)\cos u(x-t)\,dt + \int_{0}^{\infty} f(t)\cos$$

— (8)

Replacing t by $-t$, we have

$$\int_{-\infty}^{0} f(t)\cos u(x-t)\,dt = -\int_{\infty}^{0} f(-t)\cos u(x+t)\,dt$$

$$= -\int_{0}^{\infty} f(t)\cos u(x+t)\,dt \quad \text{— (9)}$$

[since $f(-t) = -f(t)$]

Using (9); equation (8) gives

$$\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt = -\int_{0}^{\infty} f(t)\cos u(x+t)\,dt + \int_{0}^{\infty} f(t)\cos u(x-t)\,dt$$

$$= \int_{0}^{\infty} f(t)\left[\cos u(x-t) - \cos u(x+t)\right] dt$$

$$= 2\int_{0}^{\infty} f(t)\sin ux \sin ut\,dt \quad — (10)$$

Substituting this in eq (7); we get

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} du \int_{0}^{\infty} f(t)\sin ux \sin ut\,dt \quad — (11)$$

changing the order of integration; we get

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} f(t)\,dt \int_{0}^{\infty} \sin ut \sin ux\,du \quad — (12)$$

Eqn (11) or (12) represents the Fourier integral for an odd function.

Case II If $f(x)$ is an even function of x i.e. $f(-x) = f(x)$; then

$$\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt = \int_{-\infty}^{0} f(t)\cos u(x-t)\,dt + \int_{0}^{\infty} f(t)\cos u(x-t)\,dt$$

Replacing t by $-t$ in first integral, we get

$$\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt = -\int_{\infty}^{0} f(-t)\cos u(x+t)\,dt + \int_{0}^{\infty} f(t)\cos u(x-t)\,dt$$

$$= \int_{0}^{\infty} f(t)\left[\cos u(x+t) + \cos u(x-t)\right] dt \quad \text{since } f(-t) = f(t)$$

$$= 2\int_{0}^{\infty} f(t)\cos ut \cos ux\,dt \quad — (13)$$

Using eqn (13), equation (7) gives

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} du \int_{0}^{\infty} f(t)\cos ut \cos ux\,dt \quad — (14)$$

Changing the order of integration; we get

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} f(t)\,dt \int_{0}^{\infty} \cos ut \cos ux\,du \quad — (15)$$

Equation (14) or (15) represents Fourier integral for an even function

Case III Complex Form of Fourier Integral

Fourier integral (4) may be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{\infty} f(t)\cos u(x-t)\,dt \quad — (16)$$

By the property of definite integrals; we also have

$$0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f(t)\sin u(x-t)\,dt \quad \text{— (17)}$$

Since $\sin u(x-t)$ is an odd function of u.

Now multiplying eqn (17) by i and adding to (1[illegible])

we get $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f(t)\, e^{iu(x-t)}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iux}\,du \int_{-\infty}^{+\infty} f(t)\, e^{-iut}\,dt \quad \text{— (18)}$$

This equation represents Fourier integral in complex form.

Putting $u = -u'$; in (18), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iu'x}\,du' \int_{-\infty}^{+\infty} f(t)\, e^{iu't}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\,du \int_{-\infty}^{+\infty} f(t)\, e^{iut}\,dt$$

This is another form of Fourier integral in complex form.

<u>Example</u>:- Find the Fourier integral of the function

$$f(x) = 0 \quad \text{when } x < 0$$
$$= \frac{1}{2} \quad \text{when } x = 0$$
$$= e^{-x} \quad \text{when } x > 0$$

Verify the representation directly at the point $x = 0$

<u>Soln</u>:- The Fourier integral of a function $f(x)$, in general, is given by

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\,dt \int_{0}^{\infty} \cos u(x-t)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} (\cos ux \cos ut + \sin ux \sin ut)\,du$$

$$= \frac{1}{\pi}\Big[\int_{0}^{\infty}\Big\{\int_{-\infty}^{+\infty} f(t)\cos ut\,dt\Big\}\cos ux\,du$$
$$+ \int_{0}^{\infty}\Big\{\int_{-\infty}^{+\infty} f(t)\sin ut\,dt\Big\}\sin ux\,du\Big]$$

Here $$\int_{-\infty}^{+\infty} f(t)\cos ut\,dt = \int_{-\infty}^{0} 0\cdot\cos ut + \int_{0}^{\infty} e^{-t}\cos ut\,dt$$

$$= \int_{0}^{\infty} e^{-t}\sin ut\,dt =$$

Now $$\int_{0}^{\infty} e^{-t}\cos ut\,dt = \int_{0}^{\infty} e^{-t}\Big\{\frac{e^{iut} + e^{-iut}}{2}\Big\}dt = \frac{1}{2}\int_{0}^{\infty} e^{-t(1-iu)} + e^{-t(1+iu)}\,dt$$

[illegible]

$\therefore \int_{-\infty} f(t)\cos ut\,dt = \frac{1}{1+u^2}$

and similarly $\int_{-\infty}^{+\infty} f(t)\sin ut\,dt = \int_0^{\infty} e^{-t}\sin ut\,dt = \frac{u}{1+u^2}$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+u^2}\cos ux\,du + \frac{1}{\pi}\int_0^{\infty} \frac{u}{1+u^2}\sin ux\,du$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos ux + u\sin ux}{1+u^2}\,du$$

Verification :- As $x = 0$, $f(x) = \frac{1}{2}$ i.e. $f(0) = \frac{1}{2}$ and from above

$$f(0) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+u^2}\,du = \frac{1}{\pi}\left[\tan^{-1}u\right]_0^{\infty} = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

Example 2 :- Express the function

$$f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

Soln :- By Fourier integral formula

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\int_{t=-\infty}^{+\infty} f(t)\cos u(x-t)\,du\,dt$$

$$= \frac{1}{\pi}\int_{u=0}^{\infty} du \int_{t=-1}^{+1} \cos u(x-t)\,dt \quad [\text{by def of function } f(x)]$$

$$= \frac{1}{\pi}\int_0^{\infty} du \left\{\frac{\sin u(x-t)}{-u}\right\}_{-1}^{+1} = \frac{1}{\pi}\int_0^{\infty}\left\{\frac{\sin u(x-1) - \sin u(x+1)}{-u}\right\}du$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{\cos ux \sin u}{u}\,du = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

This is required representation of $f(x)$ as Fourier integral

Now, we have

$$\int_0^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & |x| \le 1 \\ \frac{0}{2} & |x| > 1 \end{cases}$$

Example 3 (Laplace integrals) (a) Find the Fourier integral of

$f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ $\quad (k > 0)$

(b) Hence deduce that

$$\int_0^{\infty} \frac{\cos xu}{1+u^2} = \frac{\pi}{2}e^{-x} \quad (x > 0)$$

Soln:- Here $f(-x)=f(x)$ for $k>0$ i.e. $f(x)$ is an even function of x; hence its Fourier integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt \int_0^\infty \cos ut \cos ux \, du$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\infty \left(e^{-kt}\cos ut \, dt\right)\cos ux \, du$$

But $\int_0^\infty e^{-kt}\cos ut \, dt = \int_0^\infty e^{-kt}\left(\frac{e^{iut}+e^{-iut}}{2}\right)dt$

$$= \frac{1}{2}\int_0^\infty \left[e^{-t(k-iu)} + e^{-t(k+iu)}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-t(k-iu)}}{-(k-iu)} - \frac{e^{-t(k+iu)}}{k+iu}\right]_0^\infty$$

$$= \frac{1}{2}\left[e^{-kt}\left\{\frac{e^{iut}(k+iu)+e^{-iut}(k-iu)}{-(k^2+u^2)}\right\}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-kt}}{-(k^2+u^2)}\left\{2k\left(\frac{e^{iut}+e^{-iut}}{2}\right) - 2u\left(\frac{e^{iut}-e^{-iut}}{2i}\right)\right\}\right]_0^\infty$$

$$= \left[\frac{e^{-kt}}{k^2+u^2}\left(u\sin ut - k\cos ut\right)\right]_0^\infty = \frac{k}{k^2+u^2}$$

Therefore the Fourier integral of given function $f(x)=e^{-kx}$ is

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{k}{k^2+u^2}\cos ux \, du$$

$$= \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du \quad (\text{for } x>0,\ k>0) \quad \text{— (A)}$$

This is the required integral

(b) From (A), we have

$$\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k}e^{-kx} \quad (x>0;\ k>0)$$

Subs $k=1$, we have

$$\int_0^\infty \frac{\cos ux}{1+u^2}\,du = \frac{\pi}{2}e^{-x} \quad (x>0) \quad \text{— (B)}$$

For independent derivation of (B), consider $f(x)=e^{-x}$ and find its Fourier integral.

Example - Using Fourier integral representation show that

$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda}\sin x\lambda \, d\lambda = \begin{cases} \pi/2 & \text{when } 0<x<\pi \\ 0 & \text{when } x>\pi \end{cases}$$

Soln. The Fourier integral of a function $f(x)$, in general is given by

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut\,dt\right\}\cos ux\,du + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\left\{ f(t)\sin ut\,dt\right\}\sin ux\,du$$

$$= \frac{1}{\pi}\int_0^{\infty}\underbrace{\int_0^{\pi}\left\{\frac{\pi}{2}\cos ut\,dt\right\}\cos ux\,du}_{\to 0} + \frac{2}{\pi}\int_0^{\infty}\int_0^{\pi}\left\{\frac{\pi}{2}\sin ut\,dt\right\}\sin ux\,du$$

$$= \frac{1}{2}\int_0^{\infty}\left[\frac{-\cos ut}{u}\Big|_0^{\pi}\right]\sin ux\,du = \frac{1}{2}\int_0^{\infty}\frac{1-\cos\pi u}{u}\sin ux\,du$$

$$= \frac{1}{2}\int_0^{\infty}\frac{1-\cos\pi u}{u}\sin ux\,du.$$

<u>Fourier Integral Theorem</u> :-

The Fourier integral in complex form is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\,du\int_{-\infty}^{+\infty} f(t)\,e^{iut}\,dt$$

This identity is also known as the <u>Fourier integral theorem</u>.

<u>Theorem 1</u>:- If $f(x)$ is absolutely integrable in $(-\infty, +\infty)$ and is piecewise very smooth in every finite interval, then the Fourier integral Theorem is valid in the sense that

$$\frac{1}{\pi}\int_0^{\infty} du\int_{-\infty}^{+\infty} f(t)\cos u(x-t)\,dt = \frac{1}{2}\left[f(x+0)+f(x-0)\right]$$

<u>Theorem 2</u>:- The result of Th 1 remains valid if the statement "... is piecewise very smooth in every finite interval ..." is replaced by "... satisfies Dirichlet conditions in every finite interval ...".

We shall omit the proofs of these theorems. Because the concept of pointwise convergence is not only necessary in physics but may be even meaningless.

<u>Fourier Integral theorem</u> :-

The fourier integral in the complex form is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\,du\int_{-\infty}^{+\infty} f(t)\,e^{iut}\,dt \quad \text{— I}$$

The <u>Fourier integral theorem</u> says that, if a function $f(x)$ satisfies the Dirichlet conditions on every finite interval, and if $\int_{-\infty}^{+\infty} |f(x)|\,dx$ is finite, then ~~the function~~

(I) is correct and this integral gives the value of $f(x)$ anywhere where $f(x)$ is continuous and at jumps of $f(x)$, the integral gives the midpoint of the jump i.e. equal to $\frac{1}{2}[f(x+) + f(x-)]$.

Proof:-

* NATIONAL INFORMATICS CENTRE *