SPAN

Theo Let S be a non-empty subset of a vector space V. Then S is a subspace of V iff $span(S) = S$.

Proof Let us suppose that S is a subspace of V. Now, since $span(S)$ is the smallest subspace containing S, hence $S \subseteq Span(S)$.
Also, since S is a subspace that contains S, hence $Span(S) \subseteq S$.
So, we must have $S = Span(S)$.

Alternatively,
Let $Span(S) = S$.
Then, $span(S)$ being a subspace, S is also a subspace.

SIMPLIFIED SPAN METHOD (S.S.M)
As already taught in the class, given below are some examples on how to apply this method :-

(1) Use Simplified Span Method to find a simplified general form for all vectors in $span(S)$, where $S \subset \mathbb{R}^n$ is given by:-

$$S = \{[3, 1, -2], [-3, -1, 2], [6, 2, -4]\}$$

Now, the matrix corresponding to S is

$$\begin{bmatrix} 3 & 1 & -2 \\ -3 & -1 & 2 \\ 6 & 2 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 3 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{1}{3}R_1} \begin{bmatrix} 1 & 1/3 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\text{span}(S) = \left\{ a\left(1, 1/3, -2/3\right) : a \in \mathbb{R} \right\}$

(2) Use S.S.M to find a simplified general form for all the vectors in span(S), where S is the given subset of P_3.

$S = \{ 5x^3 + 2x^2 + 4x - 3,\ -x^2 + 3x - 7,\ 2x^3 + 4x^2 - 8x + 5,\ x^3 + 2x + 5 \}$.

Collecting all the coeff-s of x^3, x^2, x, and the constant term, form vectors in $\mathbb{R}^4$ as follows:-

$5x^3 + 2x^2 + 4x - 3 \to [5, 2, 4, -3]$

$-x^2 + 3x - 7 \to [0, -1, 3, -7]$

$2x^3 + 4x^2 - 8x + 5 \to [2, 4, -8, 5]$

$x^3 + 2x + 5 \to [1, 0, 2, 5]$

$$\begin{bmatrix} 5 & 2 & 4 & -3 \\ 0 & -1 & 3 & -7 \\ 2 & 4 & -8 & 5 \\ 1 & 0 & 2 & 5 \end{bmatrix} \xrightarrow{R_{14}} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & -1 & 3 & -7 \\ 2 & 4 & -8 & 5 \\ 5 & 2 & 4 & -3 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 5R_1]{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & -1 & 3 & -7 \\ 0 & 4 & -12 & -5 \\ 0 & 2 & -6 & -28 \end{bmatrix} \xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 4 & -12 & -5 \\ 0 & 2 & -6 & -28 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & -33 \\ 0 & 0 & 0 & -42 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{33} R_3} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -42 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 \to R_2 - 7R_3 \\ R_1 \to R_1 - 5R_3}]{R_4 \to R_4 + 42R_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now,

$$[1, 0, 2, 0] \rightarrow x^3 + 2x$$

$$[0, 1, -3, 0] \rightarrow x^2 - 3x$$

$$[0, 0, 0, 1] \rightarrow 1.$$

Hence, $\text{Span}(S) = \{ a(x^3 + 2x) + b(x^2 - 3x) + c(1) : a, b, c \in \mathbb{R} \}$

$$= \{ ax^3 + bx^2 + (2a - 3b)x + c : a, b, c \in \mathbb{R} \}$$

(3) Use SSM to find a simplified general form for all vectors in span(S), where S is the given subset of M_{22}.

$$\left\{ \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Let us first convert matrices in M_{22} into vectors in $\mathbb{R}^4$ as follows:-

$$\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \longrightarrow [1, 3, -2, 1]$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \longrightarrow [0, 0, 1, -1]$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow [-1, 0, 0, 1]$$

Now, once again we apply elementary row operations to $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

to obtain row-reduced echelon matrix.

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{R_{23}} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 3 & -2 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{3} R_2} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2/3 & 2/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + \frac{2}{3} R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Next convert each row back into the (2×2) matrices in the same way.

$$[1, 0, 0, -1] \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[0, 1, 0, 0] \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[0, 0, 1, -1] \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

Thus, Span(S) $= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

4 Prove that $S = \{[1,-2,-2], [3,-5,1], [-1,1,-5]\}$ does not span $\mathbb{R}^3$.

Proof For this, we form matrix taking each vector as a row & apply row operation to obtain row-reduced echelon form.

$$\begin{bmatrix} 1 & -2 & -2 \\ 3 & -5 & 1 \\ -1 & 1 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 7 \\ 0 & -1 & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\text{Span}(S) = \{a[1,0,12] + b[0,1,7] : a,b \in \mathbb{R}\}$

$= \{[a, b, 12a+7b] : a,b \in \mathbb{R}\}$

Now, $(1,0,1) \notin \text{Span}(S)$, but $(1,0,1) \in \mathbb{R}^3$.

Hence, $\text{Span}(S)$ cannot span $\mathbb{R}^3$.

Hence Proved.

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