Sec 2.1

Linear Transformations and Matrices

2.1 Linear Transformations, null spaces, and Ranges.

Def. Let V & W be vector space (over F). $T: V \to W$ is a linear Transformation from V to W if $\forall x, y \in V$ and $c \in F$, we have

(a) $T(x+y) = T(x) + T(y)$ and

(b) $T(cx) = cT(x)$

Remark (If F is field of rationals then (a) $\Rightarrow$ (b)).

1. If T is linear, then $T(0) = 0$ ($c=0$ in (b))
2. T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in V$ and $c \in F$.
3. If T is linear, then $T(x-y) = T(x) - T(y)$.
4. T is linear iff, $x_1, x_2, \ldots x_n \in V$ and $a_1, a_2, \ldots a_n \in F$. we have $T(\sum a_i x_i) = \sum a_i T(x_i)$

we will use (2) for proving that T is linear.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (2a_1 + a_2, a_1)$

$x, y \in \mathbb{R}^2$ & $x = (b_1, b_2)$ & $y = (d_1, d_2)$

$cx+y = (cb_1+d_1, cb_2+d_2)$

we have.

$T(cx+y) = (2(cb_1+d_1) + cb_2 + d_2,\ cb_1+d_1)$

Also $cT(x) + T(y) = c(2b_1+b_2, b_1) + (2d_1+d_2, d_1)$

$= (2cb_1 + cb_2 + 2d_1 + d_2,\ cb_1+d_1)$

$= (2(cb_1+d_1) + cb_2+d_2,\ cb_1+d_1)$

$\Rightarrow T$ is linear.

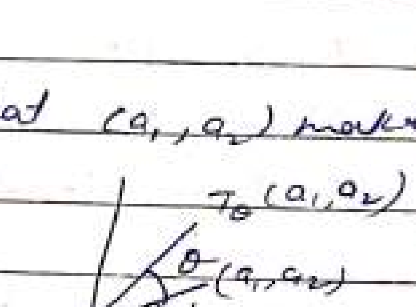
Ex. For any angle θ, define $T_\theta : R^2 \to R^2$

$T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq$ and $T_\theta(0,0) = (0,0)$. Then $T_\theta : R^2 \to R^2$ is a linear transformation that is called the rotation by θ.

Formula for T_θ -

Let $(a_1, a_2) \in R^2$

Let α be the angle that (a_1, a_2) makes with the positive x-axis



let $r = \sqrt{a_1^2 + a_2^2}$

$\Rightarrow a_1 = r\cos\alpha$ & $a_2 = r\sin\alpha$

$T_\theta(a_1, a_2)$ has length r & makes an angle $\alpha+\theta$ with x-axis

Also $T_\theta(a_1, a_2) = (r\cos(\alpha+\theta), r\sin(\alpha+\theta))$

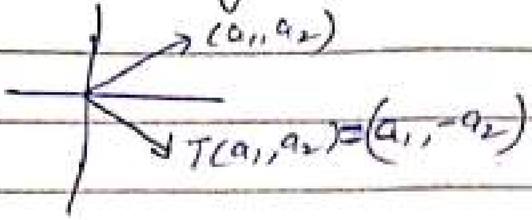
$= (r\cos\alpha\cos\theta - r\sin\alpha\sin\theta,\ r\cos\alpha\sin\theta + r\sin\alpha\cos\theta)$

$= (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$

This formula is also is valid for $(a_1, a_2) = (0,0)$

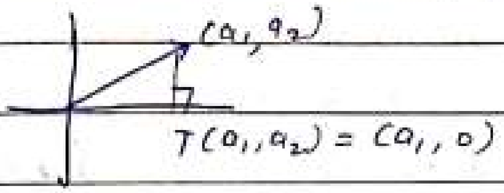
Check that T_θ is linear transformation.

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x axis.



$$T(cx+y) = T(ca_1+b_1, ca_2+b_2)$$
$$= (ca_1+b_1, -ca_2-b_2)$$
$$= c(a_1, -a_2) + (b_1, -b_2)$$

Ex Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$.
T is called the projection on the x-axis.



Ex Define $T: M_{m \times n}(F) \to M_{n \times m}(F)$ by $T(A) = A^t$

$$T(cA+B) = (cA+B)^t = cA^t + B^t = cT(A) + T(B). \quad c \in F$$

Ex $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$

$$T(ag(x)+h(x)) = (ag(x)+h(x))' = ag'(x) + h'(x)$$
$$= aT(g(x)) + T(h(x))$$

Ex Let $V = C(\mathbb{R}) \to$ v.s. of continuous real valued fns. on $\mathbb{R}$
Let $a, b \in \mathbb{R}$ & $a < b$. Define $T: V \to \mathbb{R}$ by

$$T(f) = \int_a^b f(t)\,dt$$

$$T(cf+g) = \int_a^b (cf+g) = c\int_a^b f + \int_a^b g.$$

Identity Transformation.

$I_V : V \to V$ by $I_V(x) = x \quad \forall x \in V$

& Zero Transformation

$T_0 : V \to W$ by $T_0(x) = 0. \quad \forall x \in V.$

Def: Let V and W be vector spaces, and let $T: V \to W$ be linear. Define the null space (or kernal) $N(T)$ of T to be the set of all vectors x in V such that $T(x)=0$; that is $N(T) = \{x \in V : T(x) = 0\}$

Range (or image) $R(T)$ of T is the subset of W consisting of all images (under T) of vectors in V. i.e $R(T) = \{T(x) : x \in V\}$

Ex Let V & W be vector spaces and let $I: V \to V$ & $T_0 : V \to W$ be identity & zero Transf.

Then $N(I) = \{0\}$, $R(I) = V$, $N(T_0) = V$, $R(T_0) = \{0\}$

Ex $T: R^3 \to R^2$ by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

is a L.T.

$N(T) = \{(a, a, 0) : a \in R\}$ & $R(T) = R^2$

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Thm Let V & W be V.S. & $T: V \to W$ be a linear transformation. Then $N(T)$ & $R(T)$ are subspaces of V & W, respectively.

Pr: 0_V & 0_W denote the zero vectors of V & W, respectively.

Since $T(0_V) = 0_W$

$\Rightarrow 0_V \in N(T) \Rightarrow N(T) \neq \phi$

Let $x, y \in N(T)$ & $c \in F$.

then $T(x+y) = T(x) + T(y) = 0_W + 0_W = 0_W$.

$\Rightarrow x+y \in N(T)$

& $T(cx) = cT(x) = c \cdot 0_W = 0$

$\Rightarrow cx \in N(T)$.

$\Rightarrow N(T)$ is a subspace of V.

Because $T(0_V) = 0_W$,

$\Rightarrow 0_W \in R(T) \Rightarrow R(T) \neq \phi$

Let $x, y \in R(T)$. & $c \in F$

$\Rightarrow \exists$ v & w in V s.t.

$T(v) = x$ & $T(w) = y$.

$\Rightarrow T(v+w) = T(v) + T(w) = x+y$.

$\Rightarrow x+y \in R(T)$

& $T(cv) = cT(v) = cx$

$\Rightarrow cx \in R(T)$.

$\Rightarrow R(T)$ is a subspace of W.

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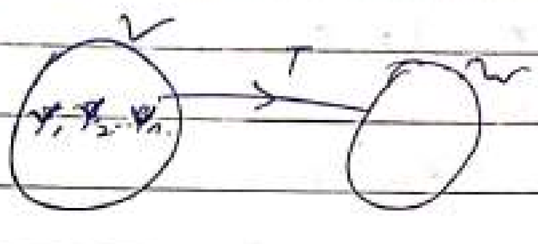
Now we will next Thm provide us method for finding spanning set for the range of a L.T from which we can find a basis for range by finding L.I set from this spanning set.

Thm: Let V & W be vector spaces, and let $T: V \to W$ be linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$$

Pr:



clearly $T(v_i) \in R(T) \quad \forall i$

$\Rightarrow \{T(v_1), T(v_2) \ldots T(v_n)\} \subseteq R(T)$

& $R(T)$ is a subspace

$\Rightarrow$ span$\{T(v_1), \ldots, T(v_n)\} \subseteq R(T)$. [if R is a subspace & $S \subseteq R$ then span S ⊆ R]

but $\{T(v_1), \ldots T(v_n)\} = T(\beta)$

$\Rightarrow$ span $T(\beta) \subseteq R(T)$.

Let $w \in R(T)$

$\Rightarrow w = T(v)$ for some $v \in V$.

Since β is basis for V

$\Rightarrow v = \sum_{i=1}^{n} a_i v_i$ for some $a_1, a_2, \ldots a_n \in F$

Since T is linear

$$T(v) = T\left(\sum a_i v_i\right)$$
$$= \sum a_i T(v_i)$$

$$w = T(v) = \sum a_i T(v_i) \in \text{span } T(\beta).$$

$$\Rightarrow R(T) \subseteq \text{span } T(\beta)$$

$$\Rightarrow R(T) = \text{span}(T(\beta)) = \text{span}\{T(v_1), T(v_2), \ldots T(v_n)\}$$

Above Thm is also true if β is infinite.

Ex. Define $T: P_2(R) \to M_{2\times 2}(R)$ by

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}.$$

Since $\beta = \{1, x, x^2\}$ is a basis for $P_2(R)$, we have

$$R(T) = \text{span}(T(\beta))$$
$$= \text{span}(\{T(1), T(x), T(x^2)\})$$
$$= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \text{ L.D.}$$
$$= \text{span}\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \text{ L.I.}$$

Thus basis for $R(T)$ is $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

So $\dim R(T) = 2$

We measure the "size" of a subspace by its dimension.

Def: Let V & W be vector spaces and let $T: V \to W$ be linear. If $N(T)$ & $R(T)$ are finite dimensional then we define the nullity of T, denoted by nullity (T), & the rank of T, denoted by rank(T) to the dimension of $N(T)$ & $R(T)$, respectively.

Intuitively we can see that larger the nullity smaller the rank & vice versa

Th. Dimension Theorem.

Let V & W be vector spaces, and let $T: V \to W$ be linear. If V is finite dimensional then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Pr: Let $\dim V = n$, $\dim(N(T)) = k$ and $\{v_1, v_2, \ldots v_k\}$ is basis for $N(T)$

$\Rightarrow \{v_1, v_2, \ldots v_k\}$ is L.I subset in V

Extend this L.I set to basis of V.

Let $\beta = \{v_1, v_2, \ldots v_n\}$ is basis for V.

claim $S = \{T(v_{k+1}), T(v_{k+2}) \ldots T(v_n)\}$ is a basis for $R(T)$

first we will prove that S generate $R(T)$

Now $T(v_i) = 0$ $\forall$ $1 \le i \le k$

because $\{v_i\}$ for $1 \le i \le k$ is basis for $N(T)$.

$\Rightarrow v_i \in N(T) \Rightarrow T(v_i) = 0$

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by ~~above Thm.~~

$R(T) = \text{span}\{T(v_1), \ldots T(v_k), \ldots T(v_n)\}$
$= \text{span}\{T(v_{k+1}), \ldots T(v_n)\}$
$= \text{span}(S)$.

Next we prove that S is L.I.

Let $\sum_{i=k+1}^{n} b_i T(v_i) = 0$ for $b_{k+1}, b_{k+2}, \ldots b_n \in F$

now T is linear

$\Rightarrow T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$ $\left\{ T \sum b_i v_i = \sum b_i T(v_i) = 0 \right\}$

$\Rightarrow \sum_{i=k+1} b_i v_i \in N(T)$.

& basis for $N(T)$ is $\{v_1, v_2, \ldots v_k\}$

$\Rightarrow \exists\, c_1, c_2, \ldots c_k \in F$ s.t.

$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$

$\Rightarrow \sum_{i=1}^{k} (-c_i) v_i + \sum_{i=k+1}^{n} b_i v_i = 0$

Since $\beta = \{v_1, v_2, \ldots v_n\}$ is basis for V

$\Rightarrow b_i = 0\ \forall i$ (also $c_i = 0$)

$\Rightarrow S$ is L.I.

~~Also $T(v_{k+1}), T(v_{k+2}) \ldots T(v_n)$ are distinct.~~ as they are L.I.

$\Rightarrow \text{Rank}(T) = n - k$

~~(If $T(v_i) = T v_j$ $i \neq j$, $k+1 \leq i, j \leq n$.~~
~~$\Rightarrow T(v_i - v_j) = 0 \Rightarrow v_i - v_j \in N(T)$.~~

$$v_i - v_j = \sum_{i=1}^{k} a_i v_i$$

$\Rightarrow a_1 v_1 + a_2 v_2 - - + a_k v_k - v_i + v_j + 0$ (sum rest of v_i's $i \leq k$)

which contradict that $\{v_1, \ldots v_n\}$ is L.I.

Ex $T : R^3 \to R^2$

$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$N(T) = \{(a, a, 0)\ a \in R\}$

& $R(T) = R^2$

$\Rightarrow$ nullity T + ran $T = 3$

$\Rightarrow$ nullity $T + 2 = 3$

$\Rightarrow$ nullity $T = 1$.

Th: Let V & W be V. spaces, and let $T : V \to W$ be linear. Then T is 1-1 iff $N(T) = \{0\}$

(i.e nullity $T = 0$ or nullity $T \{0\} = 0$)

Pr: Let T is 1-1 & $x \in N(T)$. Then

$T(x) = 0 = T(0)$.

Since T is 1-1 $\Rightarrow x = 0$.

$\Rightarrow N(T) = \{0\}$

Conv let $N(T) = \{0\}$

let $T(x) = T(y)$

$\Rightarrow T(x-y) = 0$

$\Rightarrow x - y \in N(T)$

$\Rightarrow x - y = 0$

$\Rightarrow x = y$

$\Rightarrow T$ is 1-1.

Ex. above example nullity $T = 1$
$\Rightarrow T$ is not 1-1.
(nullity of $\{0\} = 0$).

Th. Let V & W be a V.S. of equal (finite) dimension, & let $T: V \to W$ be linear. Then the following are equivalent.

(a) T is 1-1
(b) T is onto
(c) rank $T = \dim V$.

Pf. from the dimension Theorem.

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

~~from above Th. T is 1-1 iff $N(T) = \{0\}$~~
~~iff nullity $T = 0$ iff rank $T = \dim V$.~~

from above Th.

T is 1-1

$\Leftrightarrow N(T) = \{0\}$

$\Leftrightarrow$ nullity $(T) = 0$

$\Leftrightarrow$ rank $(T) = \dim V$

$\Leftrightarrow$ rank $(T) = \dim W$

$\Leftrightarrow \dim(R(T)) = \dim W$.

and since T is subspace of W.

$\Leftrightarrow R(T) = W$. $\Leftrightarrow$ T is onto.

If V is not finite dimensional and $T: V \to V$ is linear, then 1-1 & onto are not equivalent. (Exercise 15, 16 & (21))

Cone: In above 2 Thms. linearity of T is essential as we can have examples from $\underline{R}$ to $\underline{R}$ that are not 1-1 but are onto & vice versa.

?? not finite dimension.

$f: R \to R$.

$f(x) = 2x$.

Ex. $T: P_2(R) \to P_3(R)$ be a linear transformation

dim = 3   dim = 4.

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

Also:

$R(T) = \text{span}\{T(1), T(x), T(x^2)\}$.

$= \text{span}\{3x, 2 + \frac{3x^2}{2}, 4x + x^3\}$

Since $3x, 2+\frac{3x^2}{2}, 4x+x^3$ is L.I (set of poly. of diff. deg.)

rank(T) = 3.

Since dim $(P_3(R)) = 4$ T is not onto.

From dim Thm.

nullity T + 3 = 3

$\Rightarrow$ nullity T = 0 as $N(T) = \{0\}$

$\Rightarrow$ T is 1-1.

(Note) If $T: V \to W$ is 1-1 then L.In. subsets of V are mapped to L.I subsets of W & T is onto L.T. then if Span S = V then Span T(S) = W.

Ex Let $T: F^2 \to F^2$ be a linear Transf.
$T(a_1, a_2) = (a_1 + a_2, a_1)$. { dim $F^2 = 2$

clearly $N(T) = \{0\}$.
$\Rightarrow$ T is 1-1
$\Rightarrow$ T is onto (by above Th).

In Exercise 14. T is linear & 1-1 then a subset S is L.I iff T(S) is L.I.

Ex Let $T: P_2(R) \to R^3$ be a L.T.
$T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$
clearly T is linear & 1-1.

$T: V \to W$ 1-1
$\{v_1, v_2, v_3\}$ is LI in V if Tv_1, Tv_2, Tv_3 is L.I. in W

Let $S = \{2 - x + 3x^2, x + x^2, 1 - 2x^2\}$
then S is L.I in $P_2(R)$ because.
$T(S) = \{(2, -1, 3), (0, 1, 1), (1, 0, -2)\}$ is L.I in R^3.
(because T is 1-1 & onto $\Rightarrow T^{-1}$ is 1-1 & onto so L.I in R^3 ... in $P_2(R)$).

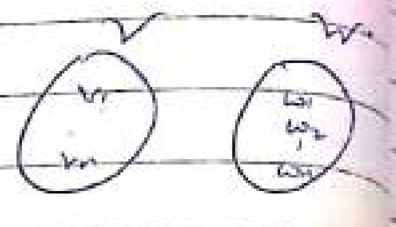
Th 6 Let V & W be vector spaces over F and suppose that $\{v_1, v_2, \dots v_n\}$ is basis for V. For $w_1, w_2, \dots w_n$ in W, there exist exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \dots n$.

Pr: Let $x \in V$. Then

$x = \sum_{i=1}^{n} a_i v_i$ where a_i's are unique scalars

Define $T : V \to W$

by $T(x) = \sum_{i=1}^{n} a_i w_i$



(a) T is linear.

Let $u, v \in V$ & $d \in F$

Then $u = \sum b_i v_i$ & $v = \sum c_i v_i$ for some scalars b_i & c_i

$\Rightarrow T(u) = \sum b_i w_i$ & $T(v) = \sum c_i w_i$

Now $\Rightarrow du + v = \sum (d b_i + c_i) v_i$

$T(du + v) = \sum (d b_i + c_i) w_i = d \sum b_i w_i + \sum c_i w_i$
$= d\,T(u) + T(v)$

(b) $T(v_i) = w_i \quad \forall\ i = 1, \dots n.$

because

$T(x) = T(\sum a_i v_i) = \sum a_i T(v_i) = \sum a_i w_i$

$\Rightarrow T(v_i) = w_i$

(c) T is unique.

Let $U : V \to W$ be linear & $U(v_i) = w_i \ \forall i = 1 \dots n$

Then for $x \in V$ with $x = \sum a_i v_i$

$U(x) = U(\sum a_i v_i) = \sum a_i U(v_i) = \sum a_i w_i = T(x)$

$\Rightarrow U = T.$

Cor Let V & W be v.s. and suppose that V has a finite basis $\{v_1, v_2, \ldots v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \ldots n$ then $U = T$.

Ex Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a L.T. s.t.

$$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$$

and suppose that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If we know that $U(1,2) = (3,3)$ and $U(1,1) = (1,3)$ then $U = T$, because $\{(1,2), (1,1)\}$ is basis for $\mathbb{R}^2$ (from above cor).

Sec 2.1 Ex 2-11, 16, 17, 21, 37, 38, 20

11 $\exists$ ~~unique~~ L.T. $T : \mathbb{R}^2 \to \mathbb{R}^3$ s.t.

~~$T(1,0) = (1,4)$ & $T(1,1) = (2,5)$~~

$T(1,1) = (1,0,2)$ & $T(2,3) = (1,-1,4)$

what is $T(8,11)$?

Is $\{(1,1), (2,3)\}$ is L.I. (i.e basis then by last thm)

~~Let $a(1,1), b(2,3) = (0,0)$~~

~~$\Rightarrow a + 2b = 0$~~

$a + 3b = 0$. So $\{(1,1), (2,3)\}$ is L.I.

$\Rightarrow$ it is basis for $\mathbb{R}^2$ (dim = 2)

$\Rightarrow b = 0$

$\Rightarrow a = 0$

So by last Th. ∃ a unique L.T.

$T: V \to W$ s.t.

$T(1,1) = (1,0,2)$

& $T(2,3) = (1,-1,4)$.

$T(8,11) = ?$

Let $(8,11) = a(1,1) + b(2,3)$.

$\Rightarrow a + 2b = 8$

$a + 3b = 11$

$\Rightarrow b = 11 - 8 = 3$

$\Rightarrow a = 8 - 2b = 8 - 6 = 2$

$\Rightarrow (8,11) = 2(1,1) + 3(2,3)$.

$\Rightarrow T(8,11) = 2T(1,1) + 3T(2,3)$

$= 2(1,0,2) + 3(1,-1,4)$

$= (5,-3,16)$

~~$T(a,b) = ? (a,b)$~~

$(a,b) = \alpha(1,1) + \beta(2,3)$

$\Rightarrow \alpha + 2\beta = a$

$\alpha + 3\beta = b$

$\Rightarrow \beta = b - a$

$\Rightarrow \alpha = a - 2\beta = a - 2(b-a)$

$= 3a - 2b$

$\Rightarrow (a,b) = (3a-2b)(1,1) + (b-a)(2,3)$

$T(a,b) = (3a-2b)T(1,1) + (b-a)T(2,3)$

$= (3a-2b)(1,0,2) + (b-a)(1,-1,4)$

$= (2a-b, a-b, 2a)$

$= (2a-b, a-b, 2a)$

$\Rightarrow T(8,11) = (16-11, 8-11, 16)$

$= (5,-3,16)$

12. Is there a L.T. ~~from~~ $T: \mathbb{R}^3 \to \mathbb{R}^2$ s.t.

$T(1,0,3) = (1,1)$ & $T(-2,0,-6) = (2,1)$?

No. Because if T is linear then

$T(-2,0,-6) = T(-2(1,0,3)) = -2T(1,0,3)$

$= -2(1,1)$

$= (-2,-2)$

$\neq (2,1)$.

13. $T: V \to W$ be linear.
If $\{w_1, \ldots, w_k\}$ is L.I subset of $R(T)$
& $S = \{v_1, \ldots, v_k\}$ s.t. $T(v_i) = w_i$ $\forall i = 1 \ldots k$.
then S is L.I.
i.e. T.S.T. if T is L.T. then
preimage of L.I subset is L.I.

Pf: Let $a_1v_1 + a_2v_2 \ldots + a_kv_k = 0$
$\Rightarrow T(\sum a_iv_i) = 0$
$\Rightarrow \sum a_i T(v_i) = 0$ (T is L.T.)
$\Rightarrow \sum a_iw_i = 0$
but $\{w_i\}$ is L.I.
$\Rightarrow a_i = 0$ $\forall i$
$\Rightarrow \{v_i\}$ is L.I.

14. Let V & W be V.S. & $T: V \to W$ be linear
(a) T is 1-1 iff T carries L.I subsets of V onto L.I subsets of W.
(b) If T is 1-1 then a subset S of V is L.I iff $T(S)$ is L.I.
(c) Suppose $\beta = \{v_1, v_2 \ldots v_n\}$ is a basis for V and T is 1-1 & onto. prove that $T(\beta)$ is basis for W.

Pr: (a) Let T is 1-1
& S is L.I subset of V
T.S.T. $T(S)$ is L.I in W.
Let $S = \{v_1, v_2 \dots v_n\}$.
Then $T(S) = \{T(v_1), T(v_2) \dots, T(v_n)\}$.
Let $\sum_{i=1}^{n} a_i T(v_i) = 0$.
$T\left(\sum a_i v_i\right) = 0 = T(0)$ (T is linear)
$\sum a_i v_i = 0$ (T is 1-1)
but $\{v_i\}$ is L.I.
$\Rightarrow a_i = 0 \ \forall i$
$\Rightarrow \{T(v_i)\}$ is L.I.

Conv: Let if S is L.I $\Rightarrow T(S)$ is L.I.
T.S.T. T is 1-1
Let T is not 1-1
$\Rightarrow N(T) \neq 0$ (both L.I)
$\Rightarrow \exists x \in N(T)$ s.t. $x \neq 0$
i.e $x \neq 0$ but $T(x) = 0$
$\Rightarrow \{x\}$ is L.I but $\{T(x)\}$ is not L.I.
(nonzero singleton is L.I & $\{0\}$ is L.D)
contradiction that T sends L.I onto L.I.
$\Rightarrow T$ is 1-1.

(b) Let $\{T(v_i)\}$ is L.I
T.S.T. $\{v_i\}$ is L.I.
Let $\sum a_i v_i = 0$
$\Rightarrow T\left(\sum a_i v_i\right) = T(0) = 0$ — No use of 1-1 here as it is same as a.13 above ??
$\Rightarrow \sum a_i T(v_i) = 0$ (T is 1-1) T is L.T.
but $\{T(v_i)\}$ is L.I.
$\Rightarrow a_i = 0 \ \forall i = 1 - n$.
$\Rightarrow \{v_i\}$ is L.I.

Conv: Let $\{v_i\}$ is L.I.
T.S.T. $\{T(v_i)\}$ is L.I
proof by (a)

(c) β is basis of V.

$\Rightarrow$ β is L.I & span $\beta = V$.

$\Rightarrow$ $T(\beta)$ is L.I. (as T is 1-1)

T.S.T. span $T(\beta) = W$.

Let $x \in W$.

$\Rightarrow \exists\ v \in V$

s.t. $T(v) = x$ (as T is onto)

$\Rightarrow v = \sum a_i v_i$ for some scalars a_i

$\Rightarrow T(v) = \sum a_i T(v_i)$

$\Rightarrow x = T(v) = \sum a_i T(v_i)$.

$\Rightarrow x \in \text{span } T(\beta)$

$\Rightarrow W \subseteq \text{Span } T(\beta)$

$\Rightarrow \text{Span } T(\beta) = W$.

15. $P(R) \rightarrow$ v.s. of poly. over R.

$T: P(R) \rightarrow P(R)$ by $T(f(x)) = \int_0^x f(t)dt$.

T.S.T T is linear & 1-1 but not onto.

Clearly $T(cf+g) = \int_0^x (cf+g) = c\int_0^x f + \int_0^x g$

If $T(f) = T(g)$

$\Rightarrow T(f-g) = 0$

$\Rightarrow \int_0^x (f-g) = 0$

$\Rightarrow f-g = 0$ (prop. of integral.

$\Rightarrow f = g.$

$\Rightarrow T$ is 1-1

<u>not onto</u>

clearly $g(x) = 1 \in P(R)$

~~but there no $f(x) \in P(R)$ s.t.~~

$\int_0^x f(t)\,dt = 1.$

So T is not onto.

16. $T : P(\mathbb{R}) \to P(\mathbb{R})$

$T(f(x)) = f'(x)$.

Clearly T is linear.

claim T is onto but not 1-1.

Let $a_0 + a_1 x + \dots a_n x^n \in P(\mathbb{R})$

then its preimage is

$a_0 x + \frac{a_1}{2} x^2 + \dots + \frac{a_n x^{n+1}}{n+1} \in P(\mathbb{R})$

so T is onto.

not 1-1

Let $T(f) = T(g)$

$\Rightarrow f' = g'$

$\Rightarrow f' - g' = 0$

$\Rightarrow (f-g)' = 0$. $(\Rightarrow f-g = \text{const.}$

$\not\Rightarrow f - g = 0$ $\quad \Rightarrow f = g + \text{const.}$

$\not\Rightarrow f = g.)$

Let $f(x) = 2$, $g(x) = 3$

then. $f' = 0$ & $g' = 0$

but $f \neq g$.

Sec 2.1

2-11-16, 17, 21, 37, 38, 20

2. $T: R^3 \to R^2$ $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$N(T) = \{(a, a, 0)\}$
$= \{a(1, 1, 0)\}$

basis $= \{(1, 1, 0)\}$

$\dim N(T) = 1 \Rightarrow \dim R(T) = 2$ $(1 + 2 = 3 = \dim V)$

$\Rightarrow$ T is not 1-1 but onto.

3. $T: R^2 \to R^3$; $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

$N(T) = \{(a_1, a_2) ; a_1 + a_2 = 0, 2a_1 - a_2 = 0\}$

$\Rightarrow a_1 = -a_2$ & $a_1 = \frac{a_2}{2}$

$\Rightarrow \frac{a_2}{2} = -a_2$

$\Rightarrow \frac{3}{2} a_2 = 0 \Rightarrow a_2 = 0$

$\Rightarrow a_1 = 0$

$N(T) = \{0\}$

$\Rightarrow$ T is 1-1

$(\dim N(T) + \dim R(T) = \dim V$
$0 + 2 = 2)$

$\Rightarrow \dim R(T) = 2 - 0 = 2$

$\Rightarrow$ T is not onto.

4. $T: M_{2\times 3} \to M_{2\times 2}$

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

$N(T)$: $2a_{11} = a_{12}$, $a_{13} = -2a_{12} = -2(2a_{11}) = -4a_{11}$

$$= \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = a_{11}\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\dim N(T) = 4$ $\Rightarrow$ T is not 1-1

$\dim R(T) = \dim V - \dim N(T) = 6 - 4 = 2$ ...

5. $T: P_2(R) \to P_3(R)$; $Tf(x) = xf(x) + f'(x)$

$\beta = \{1, x, x^2\}$,
$\gamma = \{1, x, x^2, x^3\}$

$T1 = x \cdot 1 + 0 = 0 \cdot 0 + 1 \cdot x + 0 \cdot x^2 \cdot 0x^3$

$Tx = x \cdot x + 1 = 1 + x^2 = 1 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3$

$Tx^2 = 2x + x^3 = 0 \cdot 0 + 2 \cdot x + 0 x^2 + 1 \cdot x^3$

$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$N(T)$

$xf(x) + f'(x) = 0$

$\Rightarrow f'(x) = -xf(x)$

$\Rightarrow \frac{f'(x)}{f(x)} = -x$

$\Rightarrow \ln f(x) = -\frac{x^2}{2}$

$\Rightarrow f(x) = e^{-\frac{x^2}{2}} \notin P_2(R)$

$\text{span}\, T(\beta) = R(T)$

$T(\beta) = \{x, 1+x^2, x^3+2x\}$

is L.I.

~~so rank~~
so Rank $T = 3$

$\Rightarrow N(T) = \{0\}$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow \dim R(T) = \cancel{2-0} = 3 \neq \dim P_3(R)$

$\Rightarrow T$ is 1-1 but not onto.

6. $T: M_{n \times n} \to F$

$T(A) = tr(A)$

$tr(A) = 0$

$\Rightarrow \sum_{i=1}^{n} a_{ii} = 0$

$\Rightarrow a_{nn} = -\sum_{i=1}^{n-1} a_{ii}$

$A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & -a_{11}-a_{22}-\cdots-a_{n-1\,n-1} \end{pmatrix}$

$\Rightarrow$ basis $= E^{11} - E^{nn}, E^{22} - E^{nn}, \ldots E^{n-1\,n-1} - E^{nn}, E^{12}, \ldots E^{1n}, E^{21}, E^{23} \ldots$

$\dim N(T) = \cancel{n-1}\ n^2 - 1$

$\Rightarrow \dim R(T) = n^2 - (n^2 - 1) = 1 = \dim F$

$\Rightarrow T$ is not 1-1 but onto

7.

10. $T: R^2 \to R^3$

2.1

②

10. $T: R^2 \to R^2$

$T(1,0)=(1,4)$ & $T(1,1)=(2,5)$

$(x,y) = a(1,0)+b(1,1)$

$\Rightarrow a+b=x$

$b=y$

$\Rightarrow a = x-y$

$\Rightarrow (x,y) = (x-y)(1,0) + y(1,1)$

$\Rightarrow$ ~~[illegible]~~

$\Rightarrow T(x,y) = (x-y)T(1,0) + yT(1,1)$

$= (x-y)(1,4) + y(2,5)$

$= (x-y+2y, 4x-4y+5y)$

$= (x+y, 4x+y)$

$\Rightarrow T(2,3) = (5, 4\cdot 2+3) = (5,11)$

as T is 1-1

Let $T(x,y) = T(a,b)$

$\Rightarrow (x+y, 4x+y) = (a+b, 4a+b)$ OR $N(T) = \{(x,y) \mid T(x,y)=(0,0)\}$

$\Rightarrow x+y = a+b$ $= \{(x,y) \mid (x+y, 4x+y) = (0,0)\}$

$4x+y = 4a+b$.

$\Rightarrow 3x + (a+b) = 4a+b$ $= \{(x,y) \mid x+y=0, 4x+y=0\}$

$\Rightarrow 3x = 3a$ $= \{(0,0)\}$.

$\Rightarrow x = a$

$\Rightarrow y = b$

$\Rightarrow (x,y) = (a,b)$

$\Rightarrow$ T is 1-1

11—16 in copy.

2.1 **17.** $T: V \to W$ $\dim V = n$, $\dim W = m$

(a) If $n < m$, T cannot be onto

$\dim R(T) + \dim N(T) = \dim V = n.$

$\Rightarrow \dim R(T) = n - \dim N(T) \le n < m = \dim W.$

$\Rightarrow R(T) \ne W$

$\Rightarrow T$ cannot be onto

(b) If $n > m$ T cannot be 1-1

$\dim N(T) = \dim V - \dim R(T)$
$= n - \dim R(T)$
$\ge n - \dim W = n - m$
~~$= n-m$~~.

($\dim(R(T)) \le \dim W$
$-\dim R(T) \ge -\dim W$

$\Rightarrow \dim N(T) \ge n - m > 0$

$\Rightarrow \dim N(T) \ne 0$

$\Rightarrow N(T) \ne \{0\}$

$\Rightarrow T$ can't be 1-1.

20 $T: V \to W$ is linear

V_1 ~~SD~~ & W_1 ~~SD~~ are subsp. of V & W resp

T.S.T $T(V_1)$ is subsp. of W and $T^{-1}(W_1) = \{x \in V; T(x) \in W_1\}$ is subsp. of V.

clearly $T(V_1) = \{T(x); x \in V_1\}$ is a subset of W

Let $x_1, x_2 \in T(V_1)$

$\Rightarrow x_1 = T(v_1)$, $x_2 = T(v_2)$ for some $v_1, v_2 \in V_1$

$\Rightarrow x_1 + x_2 = T(v_1) + T(v_2) = T(v_1 + v_2) \in T(V_1)$ as V_1 is a subsp.

$\cdot\ a x_1 = a T(v_1) = T(a v_1) \in T(V_1)$

$\Rightarrow T(V_1)$ is subsp. of W.

clearly $T^{-1}(W_1)$ is a subset of V.

Let $v_1, v_2 \in T^{-1}(W_1)$

$\Rightarrow$ ~~$x_1 = T($~~ $T(v_1) + T(v_2) \in W_1$

Let $T(v_1) = w_1$ & $T(v_2) = w_2$

$\Rightarrow T(v_1 + v_2) = w_1 + w_2 \in W_1$

$\Rightarrow v_1 + v_2 \in T^{-1}(W_1)$

||^{ly} $T(a v_1) = a T(v_1) \in W_1$

$\Rightarrow a v_1 \in T^{-1}(W_1)$

$\Rightarrow T^{-1}(W_1)$ is subsp. of V

Sec 2-1

1

21 V is V.S. of sequences.

$T, U : V \to V$ s.t.

$T(a_1, a_2, \dots) = (a_2, a_3, \dots)$ & $U(a_1, a_2 \dots) = (0, a_1, a_2, \dots)$

left shift right shift

let $x = (a_1, a_2 \dots)$ & $y = (b_1, b_2 \dots)$

① T is linear Ily we can prove for U.

$T(x+y) = T(a_1+b_1, a_2+b_2 \dots)$

$= (a_2+b_2, a_3+b_3 \dots)$

$= \cancel{T(x) + T(y)}$

$= (a_2, a_3 \dots) + (b_2, b_3, \dots)$

$= T(x) + T(y)$

$T(ax) = (aa_2, aa_3, \dots)$

$= a(a_2, a_3, \dots)$

$= aT(x)$

⇒ T is linear

② T is onto but not 1-1

let $x = (a_1, a_2, \dots) \in V$

$\cancel{\Rightarrow}$ $(1, a_1, a_2, \dots) \in V$

& $T(1, a_1, a_2, \dots) = (a_1, a_2 \dots)$

⇒ T is onto

not 1-1

$T(1, a_1, a_2, \dots) = T(0, a_1, a_2 \dots) = (a_1, a_2, \dots)$

but $(1, a_1, a_2 \dots) \neq (0, a_1, a_2 \dots)$

(3) U is 1-1 but not onto.

Let $U(a_1, a_2 ..) = U(b_1, b_2 ..)$

$\Rightarrow (0, a_1, a_2 ..) = (0, b_1, b_2 ..)$

$\Rightarrow (a_1, a_2 ..) = (b_1, b_2 ..)$

$\Rightarrow U$ is 1-1

not onto

For $(a_1, a_2 .. a ..) \in V$, $a_1 \neq 0$

there is no preimage in V. under T.

37 $T(x+y) = T(x) + T(y)$

T.S.T If V is a V.s over rationals, then T is linear.

~~T(ax) =~~ T.S.T $T(ax) = aT(x)$ $a \in Q$.

(1) a is ~~real~~ no. +ive integer

$T(ax) = T(x + ... x \text{ a times})$

$= Tx + Tx + .. Tx$ a times

$= a\,Tx$.

(2) a is -ive integer.

$T(ax) = T((-a)(-x))$

$= T((-x) + (-x) ... (-x) \text{ -a times})$

$= T(-x) + T(-x) -- + T(-x)$

$= (-a)\,T(-x)$

Now $T(x + (-x)) = T0 = 0$

$\Rightarrow Tx + T(-x) = 0$

$\Rightarrow T(-x) = -Tx$.

$= (-a)(-1)Tx = aTx$..

a is rational no. say $a = \frac{r}{s}$

$$T(ax) = T(\tfrac{r}{s}x) = T(ry) \quad y = \tfrac{1}{s}x \in V$$
$$= rT(y)$$
$$= rT(\tfrac{x}{s})$$

Now $T(x) = T(s \cdot \frac{x}{s})$
$$= sT(\tfrac{x}{s})$$

$\Rightarrow T(\frac{x}{s}) = \frac{1}{s}T(x)$

$\Rightarrow T(ax) = rT(\frac{x}{s}) = \frac{r}{s}T(x) = aT(x)$.

38 $T : C \to C$, $T(z) = \bar{z}$

T.S.T. T is additive i.e $T(x+y) = Tx + Ty$
but T is not linear.

$$T(z_1+z_2) = T(a+ib + c+id)$$
$$= T(a+c+i(b+d))$$
$$= a+c-i(b+d)$$
$$= a-ib+c-id$$
$$= T(a+ib) + T(c+id)$$

but
$T(iz) = T(i(a+ib)) = T(-b+ia)$
$$= -b-ia = -(b+ia)$$
$$= -i(a-ib) = \bar{i}\,\bar{z}$$

& $iTz = i\bar{z} = i(a-ib)$