Power Series

Def 1.1. A series of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

(where $c, a_0, a_1, \ldots a_n, \ldots$ are real numbers)

is called a power series around $x = c$

Def 1.2 Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, We define the radius of cgt of the power series as follows:

(i) If $\lim\sup |a_n|^{1/n} = 0$, then we define radius of cgt, $R = \infty$

(ii) If $\lim\sup |a_n|^{1/n} = \infty$, then we define radius of cgt, $R = 0$.

(iii) If $\lim\sup |a_n|^{1/n} = \frac{1}{R}$ $(0 < r < \infty)$, then radius of cgt is R.

1.3 Cauchy – Hadamard Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series (where a_n are real numbers $\forall\, n = 0, 1, 2, \ldots$)

(i) If $\lim\sup |a_n|^{1/n} = 0$, then series $\sum a_n x^n$ cgs absolutely for every x

(ii) if $\lim\sup |a_n|^{1/n} = \infty$, then series dgt for every $x \neq 0$

(iii) If $\lim_n \sup |a_n|^{1/n} = \frac{1}{R}$ $(\cancel{R \neq 0/\infty})$ $(0 < R < \infty)$

thus the series is absolutely cgt if $|x|<R$ and dgt if $|x|>R$.

Pf: (i) Suppose $\underset{n}{\text{lt}} \sup |a_n|^{1/n} = 0$.

Now $\underset{n}{\text{lt}} \sup |a_n x^n|^{1/n} = \underset{n}{\text{lt}} \sup \left(|x| \, |a_n|^{1/n}\right)$

$= |x| \underset{n}{\text{lt}} \sup |a_n|^{1/n} = |x|.0 = 0$

Thus there exist a positive integer N such that

$$|a_n x^n|^{1/n} < 1/2 \quad \forall \; n \geq N$$

ie $$|a_n x^n| < \left(\frac{1}{2}\right)^n \quad \forall \; n \geq N$$

Since geometric series $\sum \left(\frac{1}{2}\right)^n$ is cgt, by comparison test the series

$\sum a_n x^n$ is absolutely cgt for every x.

(ii) Suppose $\underset{n}{\text{lt}} \sup |a_n|^{1/n} = \infty$.

For any $x \neq 0$,

$$\underset{n}{\text{lt}} \sup |a_n x^n|^{1/n} = |x| \underset{n}{\text{lt}} \sup |a_n|^{1/n} = \infty.$$

Thus $|a_n x^n| > 1$ for infinitely many value of n

Hence $\underset{n \to \infty}{\text{lt}} a_n x^n \neq 0$, for every $x \neq 0$.

Thus the series $\sum a_n x^n$ diverges for every $x \neq 0$.

Case (iii) . Suppose $0 < R < \infty$ and $\limsup |a_n|^{1/n} = \frac{1}{R}$

For $0 < |x| < R$, $\exists$ $0 < k < 1$ such that $|x| < kR$. —— (1)

Now $\limsup_n |a_n x^n|^{1/n} = \limsup_n (|x| |a_n|^{1/n}) = |x| \limsup_n |a_n|^{1/n}$

$= \frac{|x|}{R} < k$ by (1)

Thus $\exists$ a positive integer N s.t.

$$|a_n x^n|^{1/n} < k \quad \forall n \geq N$$

$$\text{i.e } |a_n x^n| < k^n \quad \forall n \geq N$$

Since $0 < k < 1$, the geometric series $\sum k^n$ is cgt.

Hence by comparison test the series

$$\sum a_n x^n$$

is absolutely cgt.

Let $|x| > R$, Now

$$\limsup_n |a_n x^n|^{1/n} = |x| \limsup_n |a_n|^{1/n} = \frac{|x|}{R} > 1$$

$\therefore$ $|a_n x^n| > 1$ for infinitely many value of n.

Thus Seq $\{a_n x^n\}$ does not cgs to 0. Hence the series $\sum a_n x^n$ is dgt for $|x| > R$

Def let R be the radius of cgt of the power series $\sum a_n x^n$. The open interval $(-R, R)$ is called interval of cgt.

Note: The power series $\sum a_n x^n$ is absolutely cgt inside the interval of cgt & dgt outside the interval of cgt. The series may be cgt or dgt on the interval of cgt.

Problem Find radius of cgt, interval of cgt of the power series

(i) $\sum x^n$

(ii) $\sum \frac{x^n}{n}$

(iii) $\sum \frac{x^n}{n^2}$

(iv) $\sum n x^n$

(v) $\sum \frac{(-1)^{n+1} x^n}{n}$

(vi) $\sum (-1)^{n+1} \frac{x^n}{n^2}$

Also discuss ~~its~~ their cgs.

Pf: (i) $\sum x^n$

$a_n = 1$

$\therefore \underset{n}{\text{lt sup}} |a_n|^{1/n} = \underset{n}{\text{lt sup}}\, 1 = 1$

$\therefore$ radius of cgt is 1 & interval of cgt is $(-1, 1)$.

The series $\sum x^n$ is absolutely cgt for $|x| < 1$

& dgt for $|x|>$

For $x=1$, the series $\sum x^n = \sum 1$ which is dgt

$x=-1$, the series $\sum x^n = \sum(-1)^n$ which is dgt
(oscillate finitely)

(ii) $\sum \frac{x^n}{n}$

$a_n = \frac{1}{n}$.

Thus radius of cgt R is given by

$$\frac{1}{R} = \underset{n}{lt} \sup |a_n|^{1/n} = \underset{n}{lt} \sup \frac{1}{n^{1/n}} = 1$$

$\therefore R = 1$

radius of cgt is 1 & interval of cgt is $(-1, 1)$

The series $\sum \frac{x^n}{n}$ is absolutely cgt for $|x|<1$
& dgt for $|x|>1$.

For $x=1$, the series becomes $\sum \frac{1}{n}$ is dgt

For $x=-1$, the series becomes $\sum \frac{(-1)^n}{n}$ is is cgt
by Liebnitz test.

(iii) $\sum \frac{x^n}{n^2}$

$a_n = \frac{1}{n^2}$.

The radius of cgt R is given by

$$\frac{1}{R} = \underset{n}{lt} \sup |a_n|^{1/n} = \underset{n}{lt} \sup \frac{1}{(n^{1/n})^2} =$$

$$= 1 \quad (\text{Since } \underset{n}{lt}\, n^{1/n} = 1)$$

Thus radius of cgt R = 1,

& Interval of cgt is $(-1, 1)$

The power series $\sum \frac{x^n}{n^2}$ is absolutely cgt for $|x|<1$ & dgt for $|x|>1$.

For $x=1$, the series becomes $\sum \frac{1}{n^2}$. This series is cgt

For $x=-1$, the series $\sum \frac{x^n}{n^2} = \sum \frac{(-1)^n}{n^2}$ which is cgt by Liebnitz test

(iv) $\sum n x^n$

$a_n = n$

~~Since The radius of cgt~~ / R,

$\lim\sup |a_n|^{1/n} = \lim\sup n^{1/n} = 1$ $\left(\text{Since } \lim_{n\to\infty} n^{1/n} = 1\right)$

$\therefore$ Radius of cgt R is given by

$$\frac{1}{R} = \lim\sup |a_n|^{1/n} = 1$$

ie $R = 1$

Hence radius of cgt is 1 & interval of cgt is $(-1, 1)$

By Cauchy Hadamard Th the power series $\sum n x^n$ is absolutely cgt for $|x|<1$ & dgt for $|x|>1$.

For $x=1$, the series $\sum n x^n = \sum n$ is dgt

For $x=-1$, the series $\sum n x^n = \sum (-1)^n n$ is not cgt

(v) $\sum (-1)^{n+1} \frac{x^n}{n}$

$a_n = (-1)^{n+1} \frac{1}{n}$

$$\lim_n \sup |a_n|^{1/n} = \lim_n \sup \left|(-1)^{n+1}\frac{1}{n}\right|^{1/n} = \lim_n \sup\left(\frac{1}{|n|^{1/n}}\right) = 1$$

$\therefore$ radius of cgt $R = 1$ & interval of cgt is $(-1, 1)$
open interval

By Cauchy Hadamard Th, the power series $\sum (-1)^{n+1}\frac{x^n}{n}$ is absolutely cgt for $|x|<1$ & dgt for $|x|>1$.

For $x=1$, the series is

$$\sum (-1)^{n+1}\frac{x^n}{n} = \sum (-1)^{n+1}\frac{1}{n}$$ is cgt by Liebnitz test

For $x=-1$, the series becomes

$$\sum (-1)^{n+1}\frac{x^n}{n} = \sum (-1)^{n+1}\frac{(-1)^n}{n} = -\sum\frac{1}{n}$$ ~~which is~~

This series is dgt.

Note (i) If $\lim_{n\to\infty}|a_n|^{1/n}$ exists, then ~~$\lim\sup|a_n|$~~

$$\lim \sup |a_n|^{1/n} = \lim_{n\to\infty}|a_n|^{1/n}$$

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, then

$$\lim_n \sup |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|$$

~~Note If $\lim|a_n|^{1/n}$ exists~~

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Find radius of cgt & interval of cgt for series $\sum a_n x^n$, where a_n is given by

(i) $\frac{1}{n^n}$

(ii) $\frac{n^\alpha}{n!}$ (α is real number)

(iii) $\frac{n^n}{n!}$

(iv) $(\log n)^{-1}$, $n \geq 2$

(v) $\frac{(n!)^2}{(2n)!}$

(vi) $n^{-\sqrt{n}}$

Pf: (i) $a_n = \frac{1}{n^n}$

$$\text{Lt sup} |a_n|^{1/n} = \text{lt} \sup_n \left|\frac{1}{n^n}\right|^{1/n} = \text{lt sup} \frac{1}{n} = 0$$

$\therefore$ radius of cgt $R = \infty$

& interval of cgt is $(-\infty, \infty)$

(ii) $a_n = \frac{n^\alpha}{n!}$

$$a_{n+1} = \frac{(n+1)^\alpha}{(n+1)!}$$

Thus $\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha}{(n+1)!} \cdot \frac{n!}{n^\alpha} = \left(\frac{n+1}{n}\right)^\alpha \cdot \frac{1}{(n+1)}$

$= \left(1+\frac{1}{n}\right)^\alpha \frac{1}{(n+1)}$

Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^\alpha \frac{1}{(n+1)} = 0$

∴ radius of cgt $R = \infty$ & interval of cgt is $(-\infty, \infty)$.

(iv) $a_n = (\log n)^{-1} = \frac{1}{(\log n)}$

$a_{n+1} = \frac{1}{\log(n+1)}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\log n}{\log(n+1)} = 1$

(Note: By L'Hopital rule
$\lim_{x\to\infty} \frac{\log x}{\log(x+1)} \quad \frac{\infty}{\infty}$
$= \lim_{x\to\infty} \frac{1/x}{\frac{1}{x+1}}$
$= \lim_{x\to\infty} \left(1+\frac{1}{x}\right) = 1$)

Then radius of cgt $R = 1$

& interval of cgt is $(-1, 1)$

(v) $a_n = \frac{(n!)^2}{(2n)!}$

$a_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{\frac{[(n+1)!]^2}{(2n+2)!}}{\frac{n!\, n!}{(2n)!}} = \frac{(n+1)!\,(n+1)!\,(2n)!}{(2n+2)(2n+1)(2n)!\, n!\, n!}$

$= \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$

Thus

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n})}{(2+\frac{2}{n})(2+\frac{1}{n})} = \frac{1}{2.2} = \frac{1}{4}$$

$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$

$\therefore$ Radius of cgt R is given by

$$\frac{1}{R} = \lim_{n\to} \left|\frac{a_{n+1}}{a_n}\right| = 1/4$$

Thus radius of cgt $R = 4$ & interval of cgt is the open interval $(-4, 4)$