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# Series

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**Comparison Test** : Let $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$ be two real sequences and suppose that for some $K \in \mathbb{N}$,

$$0 \le x_n \le y_n \quad \forall\ n \ge K. \text{ Then}$$

(a) the convergence of $\langle y_n \rangle$ implies the convergence of $\langle x_n \rangle$

(b) the ~~convergence~~ divergence of $\langle x_n \rangle$ implies the divergence of $\langle y_n \rangle$

Proof: (a) Let $\Sigma y_n$ converges

Let $\epsilon > 0$ be given.

$\therefore$ By cauchy criterion, $\exists\ M \in \mathbb{N}$ such that

$$y_{n+1} + y_{n+2} + \text{—} + y_m < \epsilon \quad \forall\ m > n \ge M$$

Let $M' = \max\{M, K\}$

$$\therefore\ 0 < x_{n+1} + x_{n+2} + \text{—} + x_m \le y_{n+1} + \text{—} + y_m$$

$$< \epsilon \quad \forall\ m > n \ge$$

$\therefore$ By cauchy criterion $\Sigma x_n$ converges.

(b) Let $\Sigma x_n$ diverges

Let, if possible, $\Sigma y_n$ converges then by part (a) $\Sigma x_n$ converges.

a contradiction

$\therefore$ $\Sigma y_n$ diverges.

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**Thm** <u>Limit Comparison Test</u> Let $X = \langle x_n \rangle$ and $Y = \langle$ be strictly positive sequences and suppose that following limit exist in $\mathbb{R}$.

$$r = \lim\left(\frac{x_n}{y_n}\right)$$

(a) If $r \neq 0$ then $\sum x_n$ converges if and only if $\sum y_n$ converges.

(b) If $r = 0$ then if $\sum y_n$ converges then $\sum x_n$

**Proof:** (a) Since $\lim\left(\frac{x_n}{y_n}\right) = r \neq 0$ exist

$\therefore \quad r > 0 \qquad$ (as $x_n, y_n > 0 \ \forall n$ $\Rightarrow r \geq 0$ but $r \neq 0$)

$\therefore$ For $\epsilon = \frac{r}{2} > 0$, $\exists$ $K \in \mathbb{N}$ such that

$$\left|\frac{x_n}{y_n} - r\right| < \epsilon \qquad \forall n \geq K$$

$$\Rightarrow \quad r - \epsilon < \frac{x_n}{y_n} < r + \epsilon \qquad \forall n \geq K$$

$$\Rightarrow \quad r - \frac{r}{2} < \frac{x_n}{y_n} < r + \frac{r}{2} \qquad \forall n \geq K$$

$$\Rightarrow \quad \frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} \qquad \forall n \geq K.$$

$$\Rightarrow \quad \frac{r}{2} y_n < x_n < \frac{3r}{2} y_n \qquad \forall n \geq K.$$

If $\sum y_n$ converges then $\sum \frac{3r}{2} y_n$ converges

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$\therefore$ By comparison test $\sum x_n$ converges.

If $\sum x_n$ converges then by comparison test

$\sum \frac{r}{2} y_n$ converges $\Rightarrow \sum y_n$ converges.

Thus $\sum x_n$ converges $\Leftrightarrow \sum y_n$ converges

(b) If $r=0$ then $\lim \frac{x_n}{y_n} = 0$.

$\therefore$ For $\epsilon = 1 > 0$ $\exists$ $K \in \mathbb{N}$ such that

$\left| \frac{x_n}{y_n} - 0 \right| < \epsilon \qquad \forall n \geq K.$

$\Rightarrow \quad 0 < \frac{x_n}{y_n} < 1 \qquad \forall n \geq K.$

$\Rightarrow \quad 0 < x_n < y_n \qquad \forall n \geq K$

$\therefore$ By comparison test if $\sum y_n$ converges

then $\sum x_n$ converges.

Examples (a) Check the convergence of series.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

Sol: Since $0 < \frac{1}{n^2+n} < \frac{1}{n^2} \quad \forall n \geq 1.$

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and $\sum \frac{1}{n^2}$ is convergent (by p-test: $\sum \frac{1}{n^p}$ iff $p > 1$)

$\therefore$ By comparison test $\sum \frac{1}{n^2+n}$ converges.

(b) Check the convergence of $\sum \frac{1}{n^2-n+1}$

Sol. Let $x_n = \frac{1}{n^2-n+1}$ $\forall n$.

$\approx \frac{1}{n^2}$ (i.e. $x_n$ is approxim. equal to $\frac{1}{n^2}$

Let $y_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left( \frac{n^2}{n^2-n+1} \right) = 1 \neq 0$$

$\therefore$ By limit comparison test $\sum x_n$ converges if $\sum y_n$ converges

Since $\sum y_n = \sum \frac{1}{n^2}$ converges by p-test.

$\therefore$ $\sum x_n$ convergent.

(c) Check the convergent of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

Sol. Let $x_n = \frac{1}{\sqrt{n+1}} \approx \frac{1}{\sqrt{n}}$

Let $y_n = \frac{1}{\sqrt{n}}$

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$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 \neq 0$$

$\therefore$ By limit comparison test $\Sigma x_n$ converges iff $\Sigma y_n$ converges.

But $\Sigma y_n = \Sigma \frac{1}{\sqrt{n}} = \Sigma \frac{1}{n^{1/2}}$ diverges (by p-test)

$\therefore$ $\Sigma x_n$ diverges.

(d) Check the convergent of $\Sigma \frac{1}{n!}$

Sol. Let $x_n = \frac{1}{n!}$ $\forall n$.

Since $n! > n^2$ $\forall n \geq 4$.

as $1! = 1$, $2! = 2$, $3! = 3.2 = 6$, $4! = 4.3! > 4^2$

$5! = 5.4! > 5^2$

$\vdots$

$\Rightarrow$ $\frac{1}{n!} < \frac{1}{n^2}$ $\forall n \geq 4$

~~$\therefore$ By comparison test~~

Let $y_n = \frac{1}{n^2}$

Since $\Sigma y_n$ converges by p-test

$\therefore$ By comparison test $\Sigma x_n$ converges.

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## Absolute Convergence

**Def.** A series $\sum x_n$ is said to be absolutely convergent if the series $\sum |x_n|$ convergent in R.

**Def.** A series $\sum x_n$ is said to be conditionally convergent if $\sum x_n$ is convergent but it is not absolutely convergent.

**Theorem:** If a series $\sum x_n$ is absolutely convergent then it is convergent.

**Proof:** Let $\sum x_n$ is absolutely convergent.

$\Rightarrow$ $\sum |x_n|$ convergent.

$\therefore$ By Cauchy criterion, for given $\epsilon > 0$ $\exists$ $M \in \mathbb{N}$ such that.

$$|x_{n+1}| + |x_{n+2}| + \text{---} + |x_m| < \epsilon \quad \forall\, m > n \geq M$$

Now by triangular inequality.

$$|x_{n+1} + \text{---} + x_m| \leq |x_{n+1}| + |x_{n+2}| + \text{---} + |x_m|$$

$$< \epsilon \quad \forall\, m > n \geq M.$$

$\therefore$ By Cauchy criterion

$\sum x_n$ converges.

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# Grouping of series.

**Thm** If series $\Sigma x_n$ converges then any series obtained from it by grouping the term is also convergent and to the same sum.

**Proof:** Let $\Sigma y_n$ be grouping series of $\Sigma x_n$.

Let $y_1 = x_1 + x_2 + \cdots + x_{k_1}$, $y_2 = x_{k_1+1} + \cdots + x_{k_2}$, and so on

If $\langle S_n \rangle$ be S.O.P.S. of $\Sigma x_n$

and $\langle T_k \rangle$ be S.O.P.S of $\Sigma y_k$.

$\therefore$ we have $T_1 = y_1 = S_{k_1}$

$T_2 = y_1 + y_2 = S_{k_2}$

$\vdots$

$\therefore \langle T_k \rangle = \langle S_{k_i} \rangle$ is subsequence of $\langle S_n \rangle$

If $\Sigma x_n$ converges to $S$ then $\langle S_n \rangle$ converges to $S$

$\therefore$ its subsequence $\langle T_k \rangle$ also converges to $S$

$\Rightarrow \Sigma y_n$ converges to $S$.

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Remark :

But converse is not true .

For eg. series $\sum_{n=1}^{\infty}(-1)^{n+1}$ is divergent by $n^{th}$ term test as $\lim_{n\to\infty}(-1)^{n+1} \neq 0$

But its grouped series .

$$= (1-1) + (1-1) + (1-1) + \text{———}$$

$$= 0 + 0 + 0 + \text{———}$$

$$= 0$$

is convergent to 0 .

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# Test for Absolute Convergence

## Limit Comparison test II

Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two non-zero ~~series~~ sequences and $r = \lim_{n\to\infty} \left|\frac{x_n}{y_n}\right|$ exist in $\mathbb{R}$

(a) If $r \neq 0$ then $\sum x_n$ is absolutely convergent iff $\sum y_n$ is absolutely convergent

(b) If $r = 0$ and if $\sum y_n$ is absolutely convergent then $\sum x_n$ is absolutely convergent

Proof: (Proof is similar to limit comparison test)

## Theorem (Root test)

Let $\langle x_n \rangle$ be seq. in $\mathbb{R}$ and suppose that

$$r = \lim_{n\to\infty} |x_n|^{1/n} \text{ exist in } \mathbb{R}$$

then ① $\sum x_n$ is absolutely convergent if $r < 1$

② $\sum x_n$ is divergent if $r > 1$

③ Test fails if $r = 1$.

Proof: (1) let $r < 1$
then $\exists$ $r_1 \in \mathbb{R}$ such that $r < r_1 < 1$

Let $\epsilon = r_1 - r > 0$

Since $\lim_{n \to \infty} |x_n|^{1/n} = r$

$\therefore$ for above $\epsilon > 0$, $\exists$ $K \in \mathbb{N}$ such that

$$\left| |x_n|^{1/n} - r \right| < \epsilon \quad \forall\ n \geq K.$$

$\Rightarrow$ $r - \epsilon < |x_n|^{1/n} < r + \epsilon \quad \forall\ n \geq K$

$\Rightarrow$ $2r - r_1 < |x_n|^{1/n} < r_1 \quad \forall\ n \geq K$

$\Rightarrow$ $|x_n| < r_1^n \quad \forall\ n \geq K.$

Since $\sum r_1^n$ is Geometric series with common ratio $r_1 < 1$, hence converges.

$\therefore$ By comparison test $\sum |x_n|$ converges

$\Rightarrow$ $\sum x_n$ converges absolutely.

(2) If $r > 1$ then $\exists$ $K \in \mathbb{N}$ such that

$|x_n|^{1/n} > 1 \quad \forall\ n \geq K \Rightarrow |x_n| > 1 \quad \forall\ n \geq K$

$\therefore$ $\lim_{n \to \infty} |x_n| \neq 0$

$\Rightarrow$ by $n^{th}$ term, $\sum x_n$ diverges.

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(3) If $r = 1$ then test fails.

For eg $x_n = \frac{1}{n}$ then $|x_n|^{1/n} = \frac{1}{n^{1/n}} \to 1$ as $n \to \infty$

But $\Sigma |x_n|$ diverges.

If $y_n = \frac{1}{n^2}$ then $\lim |y_n|^{1/n} = \lim \frac{1}{(n^2)^{1/n}} = 1$

But $\Sigma \frac{1}{n^2}$ converges absolutely.

Theorem Ratio test

Let $\langle x_n \rangle$ be a non-zero sequence of real numbers and suppose that

$$r = \lim \left| \frac{x_{n+1}}{x_n} \right| \text{ exist in } \mathbb{R}$$

Then ① $\Sigma x_n$ is absolutely convergent if $r < 1$

② $\Sigma x_n$ is divergent if $r > 1$

③ Test fails if $r = 1$

Proof: ① let $r < 1$ then $\exists\, r_1 \in \mathbb{R}$ such that $r < r_1 < 1$. Let $\epsilon = r_1 - r > 0$

Since $\lim_{n \to \infty} \left| \frac{x_{n+1}}{x_n} \right| = r$

$\therefore$ For above $\epsilon > 0$, $\exists\, K \in \mathbb{N}$ such that

$$r - \epsilon < \left| \frac{x_{n+1}}{x_n} \right| < r + \epsilon \quad \forall\, n \geq K$$

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$$\left|\frac{x_{n+1}}{x_n}\right| < r_1 \qquad \forall\ n \geq K.$$

$$|x_{n+1}| < r_1 |x_n| \qquad \forall\ n \geq K$$

$$< r_1^2 |x_{n-1}|$$

$$\vdots$$

$$< r_1^{n-K+1} |x_K| \qquad (K = n-(n-K))$$

$$= \frac{r_1^n}{r_1^{K-1}} |x_K|$$

$$= C\, r_1^n \qquad \text{where} \quad C = \frac{|x_K|}{r_1^{K-1}}$$

$$\forall\ n \geq K.$$

Since $\sum r_1^n$ is Geometric series with common ratio $r_1 < 1$, hence converges.

$\Rightarrow$ $\sum C r_1^n$ converges.

$\Rightarrow$ By comparison test $\sum |x_n|$ converges.

$\Rightarrow$ $\sum x_n$ converges absolutely.

(2) If $r > 1$ then $\exists$ $K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| > 1 \qquad \forall\ n \geq K$$

$$|x_{n+1}| > |x_n| \qquad \forall\ n \geq K$$

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$\therefore$ $\langle |x_n| \rangle$ can not converges to 0

$\therefore$ By $n^{th}$ term test, $\sum x_n$ diverges.

(3) If $r=1$ test fails.

as if $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^2}$ then,

$$\lim \left|\frac{x_{n+1}}{x_n}\right| = \lim \frac{x_{n+1}}{x_n} = 1$$

$$\lim \left|\frac{y_{n+1}}{y_n}\right| = \lim \frac{y_{n+1}}{y_n} = 1$$

But $\sum x_n$ diverges and $\sum y_n$ converges by $p$-test.