orthogonal complements :

Def: Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W, denoted by $W^\perp$ (Read as W perp) in $\mathbb{R}^n$, is the set of all vectors $x \in \mathbb{R}^n$ with the property that $x \cdot y = 0,\ \forall\ y \in W$

$W^\perp \rightarrow$ orthogonal complement of subspace W

$$W^\perp = \{ x \in \mathbb{R}^n : x \cdot y = 0 \ \forall\ y \in W \}$$

OR

$W^\perp$ contains those vectors of $\mathbb{R}^n$ orthogonal to every vector in W

Thm (1) If W is a subspace of $\mathbb{R}^n$, then $v \in W^\perp$ if and only if v is orthogonal to every vector in a spanning set for W.

Pf: Let $S = \{ w_1, w_2, \dots, w_k \}$ be a spanning set for W

$\Rightarrow span(S) = W$

($\Rightarrow$) Let $v \in W^\perp$

T.S.T : $v \cdot w_i = 0 \quad \forall\ i = 1$ to k

For,

$v \in W^\perp \Rightarrow v \cdot y = 0 \ \forall\ y \in W$ (by def)

But $S \subseteq span(S) = W$

$\Rightarrow S \subseteq W$

$\Rightarrow v \cdot w_i = 0 \quad \forall\ i = 1$ to k

($\Leftarrow$) suppose $v \cdot w_i = 0 \quad \forall\ i = 1$ to k ...(1)

T.S.T : $v \in W^\perp$

E.T.S : $v \cdot y = 0 \quad \forall\ y \in W$

Let $y \in W = span(S)$

$\Rightarrow \exists\ a_1, a_2, \dots, a_k \in \mathbb{R}$ s.t.

$y = a_1 w_1 + a_2 w_2 + \cdots + a_k w_k$

$\Rightarrow v \cdot y = v \cdot (a_1 w_1 + a_2 w_2 + \cdots + a_k w_k)$

$= a_1 \underbrace{(v \cdot w_1)}_{=0} + a_2 \underbrace{(v \cdot w_2)}_{=0} + \cdots + a_k \underbrace{(v \cdot w_k)}_{=0}$

$(\because$ by (1))

$\Rightarrow v \cdot y = 0$

$\Rightarrow v \cdot y = 0 \quad \forall \; y \in W$

$\Rightarrow \underline{v \in W^{\perp}}$

e.g.: (1) $\mathbb{R}^n$, $W = \mathbb{R}^n$
subspace

Then $(\mathbb{R}^n)^{\perp} = \{0\}$ (How?)

$\because \mathbb{R}^n = \text{span}\{e_1, e_2, \ldots, e_n\}$

; $e_i = \begin{cases} 1 & \text{at } i\text{-th place} \\ 0 & \text{other place} \end{cases}$

$= \{0, 0, \ldots 0, \underset{i\text{-th place}}{1}, 0, \ldots 0\}$

$\{(x_1, x_2, \ldots x_n) \in (\mathbb{R}^n)^{\perp} : (x_1, x_2, \ldots x_n)$ is orthogonal to every vector in spanning set by $\mathbb{R}^n$ (1) $\}$

$\therefore x = (x_1, x_2, \ldots x_n) \in (\mathbb{R}^n)^{\perp}$ iff

$(x_1, x_2 \ldots x_n) \cdot (e_1) = (0, 0, \ldots 0)$

$\Rightarrow x_1 = 0$

$(x_1, x_2 \ldots x_n) \cdot e_2 = (0, 0, \ldots 0)$

$\Rightarrow x_2 = 0$

$(x_1, x_2, \ldots x_n) \cdot e_n = 0$

$\Rightarrow x_n = 0$

$\therefore x \in (\mathbb{R}^n)^{\perp}$ iff $x = 0$

$\therefore \underline{(\mathbb{R}^n)^{\perp} = \{0\}}$

Also, the orthogonal complement of the trivial subspace $\{0\}$ of $\mathbb{R}^n$

$$\{0\}^{\perp} = \mathbb{R}^n$$

$\because$ each vector in $\mathbb{R}^n$ is orthogonal to zero vector

Q. (2) consider the subspace $W = \{a[-1,2,3] : a \in \mathbb{R}\}$ Find $W^{\perp}$

<u>Solⁿ</u>:

$$W = \{a[-1,2,3] : a \in \mathbb{R}\}$$

$$\Rightarrow \quad W = \text{span}\{[-1,2,3]\} \quad -①$$

AIM is to find $W^{\perp}$

So $[x,y,z] \in W^{\perp}$ iff $[x,y,z]\cdot[-1,2,3] = 0$

iff $-x+2y+3z = 0$

$\therefore W^{\perp} = \{[x,y,z] \in \mathbb{R}^3 : \underbrace{[x,y,z]\cdot[-1,2,3]}_{} = 0\}$ $\Rightarrow$ span W

$= \{[x,y,z] \in \mathbb{R}^3 : -x+2y+3z = 0\}$

$\Rightarrow W^{\perp}$ is precisely the set of all vectors $[x,y,z]$ lying in the plane $-x+2y+3z=0$

Also

$$-x+2y+3z = 0 \quad — (*)$$

choose $y=1, z=0$	$y=0, z=1$
$\Rightarrow x = 2$	$\Rightarrow x = 3$
$\Rightarrow [2,1,0]$	$\Rightarrow [3,0,1]$

two linearly independent solⁿ of eqⁿ (*)

$\therefore W^{\perp}$ is spanned by two linearly independent vectors $[2,1,0]$ & $[3,1,0]$

$\Rightarrow W^{\perp} = \text{span}\{[2,1,0],[3,1,0]\}$ — ②

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Also $W^\perp$ is a subspace of $\mathbb{R}^3$

$\therefore$ $\dim W^\perp = 2$ (by eq^n (2))
$\dim W = 1$ (by eq^n (1))

$\Rightarrow \dim W + \dim W^\perp = 1 + 2 = 3 = \dim \mathbb{R}^3$

Q.3 consider the subspace
$W = \{(a, 0, c) : a, c \in \mathbb{R}\}$ of $\mathbb{R}^3$
Then find $W^\perp$. Also verify $\dim W + \dim W^\perp$
$= \dim \mathbb{R}^3$

H.W.

Thm: Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$ and $W \cap W^\perp = \{0\}$

Pf: Let W: a subspace of $\mathbb{R}^n$.

$\because$ $W^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \ \forall\ y \in W\}$

T.S.T: (1) $W^\perp$ is a subspace of $\mathbb{R}^n$.

(2) $W \cap W^\perp = \{0\}$

(1) $W^\perp$ is a subspace of $\mathbb{R}^n$

$0 \in \mathbb{R}^n$: $0 \cdot y = 0 \ \forall\ y \in W$

$\Rightarrow 0 \in W^\perp$

$\Rightarrow W^\perp \neq \phi$ ... (1)

Next,

Let $x, y \in W^\perp$, $w_1 \in W$

$\Rightarrow x \cdot w_1 = 0 \ \forall\ w_1 \in W$ ... (2)

& $y \cdot w_1 = 0 \ \forall\ w_1 \in W$

Now $(x+y) \cdot w_1 = x \cdot w_1 + y \cdot w_1$

$= 0 + 0$ (by (2))

$= 0$

$\Rightarrow x+y \in W^\perp$ ... (3)

again, let $x \in W^\perp$ & c: be a scalar

then we have to show $cx \in W^\perp$

For,

$(cx) \cdot w_1 = c(x \cdot w_1)$; where $w_1 \in W$

$= c \cdot 0$

$(cx) \cdot w_1 = 0$

$\Rightarrow cx \in W^\perp$ ... (4)

(1) + (3) + (4) $\Rightarrow W^\perp$ is a subspace of $\mathbb{R}^n$.

(2)

T.S.T: $W \cap W^\perp = \{0\}$

For,

Let $x \in W \cap W^\perp$

$\Rightarrow x \in W$ & $x \in W^\perp$

$\Rightarrow x \in W$ & $x \in W^{\perp}$
$$\Downarrow$$
$x \cdot x = 0$

$\Rightarrow \|x\|^2 = 0$

$\Rightarrow x = 0$

$\therefore W \cap W^{\perp} = \{0\}$

Th^m (x) Let W be a subspace of $\mathbb{R}^n$. Let $\{u_1, u_2, \dots, u_k\}$ be an orthogonal basis for W contained in an orthogonal basis $\{u_1, u_2, \dots, u_k, u_{k+1}, \dots, u_n\}$ for $\mathbb{R}^n$. Then $\{u_{k+1}, u_{k+2}, \dots, u_n\}$ is an orthogonal basis for $W^{\perp}$.

Cor (1) Let W be a subspace of $\mathbb{R}^n$. Then
$$\dim(W) + \dim(W^{\perp}) = n = \dim(\mathbb{R}^n)$$

Pf: Let W be a subspace of $\mathbb{R}^n$ of dim k. - (1)

Note (1) Let $B = \{w_1, w_2, \dots, w_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{u_1, u_2, \dots, u_k\}$ obtained by applying the Gram-Schmidt process to B is an orthogonal basis for W.
Hence, any non-trivial subspace W of $\mathbb{R}^n$ has an orthogonal basis

$\because$ W is a subspace of $\mathbb{R}^n$ of dim k

$\Rightarrow$ By above Note (1), the W has an orthogonal basis $\{u_1, u_2, \dots, u_k\}$

$\Rightarrow$ we have an orthogonal set

~~Note (2)~~ $\{u_1, u_2, \dots, u_k\}$ of non zero vectors in $\mathbb{R}^n$ - (2)

Note (2): Let W be a subspace of $\mathbb{R}^n$. Then any orthogonal set of non-zero vectors in W can be enlarged to an orthogonal basis for W. Similarly, any orthonormal set of vectors in W can be enlarged to an orthonormal basis for W

$\therefore$ by above Note (2), eqn (2) can be enlarged to an orthogonal basis

$$\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots u_n\} \text{ for } \mathbb{R}^n$$

$\therefore$ By thm (*)

$\{u_{k+1}, u_{k+2}, \ldots, u_n\}$ is an orthogonal basis for $W^\perp$

$$\Rightarrow \dim W^\perp = n-k$$

$$\therefore \dim W + \dim W^\perp = k + (n-k)$$
$$= n = \dim \mathbb{R}^n.$$

e.g. (1) In $\mathbb{R}^3$, the one-dimensional subspace

$$W = \text{span}\{[a,b,c]\} \; ; \; [a,b,c] \neq [0,0,0]$$

$\Rightarrow$ By cor 1, W has a two dimensional orthogonal complement

spanning set for W

$$\text{i.e. } W^\perp = \{[x,y,z] \in \mathbb{R}^3 : [x,y,z] \cdot [a,b,c] = 0\}$$
$$= \{[x,y,z] \in \mathbb{R}^3 : ax+by+cz = 0\}$$

$\therefore W^\perp$ is precisely the set of vectors $[x,y,z]$ lying in the plane $ax+by+cz=0$

KeyNotes (i) Gram-Schmidt orthogonal process.
(2) Enlarging method.

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Corr (2): Let W be a subspace of $\mathbb{R}^n$. Then $(W^\perp)^\perp = W$
H.W.

Q.1 Consider the subspace $W = \text{span}\{[2,-1,0,1], [-1,3,1,-1]\}$ of $\mathbb{R}^4$. Find an orthogonal basis for $W^\perp$

Soln:

Let $w_1 = [2,-1,0,1]$, $w_2 = [-1,3,1,-1]$
Our first aim is to find an orthogonal basis for W

Perform Gram-Schmidt process to find an orthogonal basis for W

Step (1) Let $v_1 = w_1 = [2,-1,0,1]$

$$v_2 = w_2 - \frac{w_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$$

$w_2 \cdot v_1 = [-1,3,1,-1]\cdot[2,-1,0,1] = -2-3+0-1$
$= -6$

$v_1 \cdot v_1 = 4+1+0+1 = 6$

$\therefore v_2 = [-1,3,1,-1] - \left(\frac{-6}{6}\right)[2,-1,0,1]$

$= [-1,3,1,-1] + [2,-1,0,1]$
$= [1,2,1,0]$

Set $B = \{v_1, v_2\} = \{[2,-1,0,1], [1,2,1,0]\}$
is an orthogonal basis for W

We now expand this basis B for W to a basis for $\mathbb{R}^4$ using the enlarging method.

for, let

$r = \{ u_1, u_2, e_1, e_2, e_3, e_4 \}$

$\therefore e_1 = [1,0,0,0]$
$e_2 = [0,1,0,0]$
$e_3 = [0,0,1,0]$
$e_4 = [0,0,0,1]$

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Big\downarrow$ apply Gauss-Jordan method

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$r' = \{ \overset{u_1}{[2,-1,0,1]}, \overset{u_2}{[1,2,1,0]}, \overset{e_1}{[1,0,0,0]}, \overset{e_2}{[0,1,0,0]} \}$$

$\therefore$ is a basis for $\mathbb{R}^4$.

$r' = \{ u_1, u_2, e_1, e_2 \}$

Now apply Gram-Schmidt process again we get, (Easy)

$$r'' = \{ \underbrace{\overset{u_1}{[2,-1,0,1]}, \overset{u_2}{[1,2,1,0]}}_{W}, \underbrace{\overset{u_3}{[1,0,-1,-2]}, \overset{u_4}{[0,1,-2,1]}}_{W^\perp} \}$$

A orthogonal basis for $\mathbb{R}^4$

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$$\Rightarrow V'' = \{\underbrace{u_1, u_2}_{\text{orth Basis for } W}, u_3, u_4\}$$ is an orthogonal basis for $\mathbb{R}^4$

$\therefore$ By $th^m (*)$, we get

$\{u_3, u_4\} = \{[1, 0, -1, -2], [0, 1, -2, 1]\}$ is an orthogonal basis for $W^\perp$

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