Taylor's Theorem:

If a function f defined on $[a, a+h]$ is such that

(i) the $(n-1)^{th}$ derivative f^{n-1} is continuous in $[a, a+h]$

(ii) the $(n-1)^{th}$ derivative f^{n-1} is derivable in $(a, a+h)$

then there exists some θ; $0<\theta<1$, such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $R_n = \frac{h^n}{n!} f^n(a+\theta h)$ [Lagrange's form of Remainder]

and $R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) \rightarrow$ Cauchy's Remainder [see page 63]

Proof by given hypothesis $f, f', f'', \cdots f^{n-1}$ all are continuous in $[a, a+h]$ and differentiable on $(a, a+h)$ — ①

We define a function ϕ on $[a, a+h]$ as follows.

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)^n \quad \forall x \in [a, a+h]$$ — ②

where A is constant to be determined by

$\phi(a) = \phi(a+h)$ — ③

from ② and ③ we obtain

$$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + A h^n = f(a+h)$$ — ④

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from ① and ②, we see that

ϕ is continuous in $[a, a+h]$ and ϕ is derivable in $(a, a+h)$, Also $\phi(a) = \phi(a+h)$

thus ϕ satisfies all the conditions of Rolle's theorem and so there exists some θ, $0<\theta<1$ such that

$$\phi'(a+\theta h) = 0 \quad \text{— (5)}$$

Here $(c = a+\theta h \in (a, a+h))$

differentiating ② on both sides, we obtain.

$$\phi'(x) = \cancel{f'(x)} - \cancel{f'(a)} + \cancel{(a+h-x) f''(x)} - \cancel{(a+h-x) f''(x)} + \cdots \frac{(a+h-x)^{n-1}}{(n-1)!} \cdot f^n(x) - nA(a+h-x)^{n-1}$$

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} \cdot f^n(x) - nA(a+h-x)^{n-1}$$

put $x = a + \theta h$

$$\phi'(a+\theta h) = \frac{(h-\theta h)^{n-1}}{(n-1)!} f^n(a+\theta h) - nA(h-\theta h)^{n-1}$$

$$0 = h^{n-1}(1-\theta)^{n-1}\left[\frac{1}{(n-1)!} f^n(a+\theta h) - nA\right]$$

(using ⑤) $\qquad (h \neq 0, \theta \neq 1)$

$$\Rightarrow \frac{1}{(n-1)!} f^n(a+\theta h) - nA = 0$$

$$\Rightarrow A = \frac{1}{n \cdot (n-1)!} f^n(a+\theta h) \Rightarrow \boxed{A = \frac{1}{n!} f^n(a+\theta h)}$$

Put the value of A in ④

$$\boxed{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)}$$

Maclaurin's Theorem

On taking $a=0$ in Taylor's theorem we get Maclaurin's theorem which may be stated as follows:

If f is defined on $[0,h]$ is such that

(i) f^{n-1} ($(n-1)^{th}$ derivative) is continuous in $[0,h]$

(ii) f^{n-1} is derivable in $(0,h)$

then for each x in $[0,h]$ there exist a real number θ between 0 and 1 such that

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)}$$

Maclaurin's Infinite Series

Let $S_n = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$

$R_n = \frac{x^n}{n!} f^n(\theta x)$

$\therefore$ $f(x) = S_n + R_n$, Now $R_n \to 0$ as $n \to \infty$ if and only if $f(x) = \lim_{n\to\infty} S_n$, thus we have have the following

(I) If function f defined on $[0,h]$ is such that

(I) f^n exists for each n i.e. f has derivatives of all orders.

(II) $R_n \to 0$ (R_n is tending to zero) as $n \to \infty$ then

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^n(0)}$$

This is called Maclaurin's infinite series expansion of $f(x)$.

Some Expansions

(1) Maclaurin's Series Expansion of e^x

$f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x \cdots$, $f^n(x) = e^x \ \forall x$

thus f has derivative for all n.

further $f'(0) = 1$, $f''(0) = 1$, $f'''(0) = 0$, $\cdots$ $f^n(0) = 1$ $\forall n$

The Lagrange's remainder after n terms is

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} \cdot e^{\theta x}, \quad 0 < \theta < 1$$

Now we have to show $R_n \to 0$ as $n \to \infty$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left(\frac{x^n}{n!}\right) e^{\theta x} \quad \left(\because \lim_{n \to \infty} \frac{x^n}{n!} = 0 \ \forall x\right)$$

$= 0 \cdot e^{\theta x} = 0$, both conditions of Maclaurin's series expansion are satisfied

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

Hence $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

(2) Maclaurin's Series expansions of $\sin x$

$f(x) = \sin x, \ \forall x \in \mathbb{R}$

$f'(x) = \cos x = \sin(x + \pi/2), \ f'(0) = 1$

$f''(x) = -\sin x = \sin(x + 2 \cdot \frac{\pi}{2}) = \sin(x+\pi)$

$f'''(x) = -\cos x = \sin(x + 3 \cdot \frac{\pi}{2})$

$\vdots$

$f^n(x) = \sin(x + \frac{n\pi}{2}) \ \forall n \in \mathbb{N}$

Thus f has derivative of all orders.

Now $|f^n(\theta x)| = |\sin(\theta x + \frac{n\pi}{2})| \leq 1 \ \forall n \ \& \ \forall x \in \mathbb{R}$

$|R_n| = |\frac{x^n}{n!} f^n(\theta x)| \leq |\frac{x^n}{n!}| \longrightarrow 0$ as $n \to \infty$

Thus $R_n \longrightarrow 0$ as $n \to \infty$ and so by Maclaurin's series

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-1) + \cdots$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

(3) Similarly $\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$

(Do Yourself)