Section 9.4 (Series of functions)

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:- $\sum_{n=1}^{\infty} f_n$ = series of functions

$$f_n : D \subseteq \mathbb{R} \to \mathbb{R}$$

For each $x \in D$

$\sum f_n(x)$ is a series of real number.

- $\langle S_n \rangle$ is a series of partial sum where

$$S_n = x_1 + x_2 + x_3 + \cdots + x_n$$

&

$$l = \sum_{n=1}^{\infty} x_n$$

OR

$\langle S_n \rangle \to$ sequence of partial sums

where $S_n = f_1 + f_2 + \cdots + f_n$

and each S_n is a function s.t. $S_n : D \longrightarrow \mathbb{R}$

For each $x \in D$, $f(x) = l(x) = \lim S_n(x) \;\; \forall x \in D$

$$\sum f_n(x) = f(x)$$

i.e. $\sum f_n \to f$

Definition: 9.4.1 :- If $\langle f_n \rangle$ is a seq^n of funct^n defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the seq^n of partial sums $\langle S_n \rangle$ of the infinite series $\sum f_n$ is defined for x in D by

$$S_1(x) = f_1(x)$$

$$S_2(x) = S_1(x) + f_2(x) = f_1(x) + f_2(x)$$

$$S_{n+1}(x) = S_n(x) + f_{n+1}(x) = f_1(x)+f_2(x)+\cdots+f_{n+1}(x)$$

In case, the seq^n $\langle S_n \rangle$ of functions converges on D to a function f, we say that the infinite series of function $\sum f_n$ converges to f on D.

* Further if the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is absolutely convergent on D.

* If the seq^n $\langle S_n \rangle$ of partial sums is uniformly cgt on D to f, we say that $\sum f_n$ is uniformly convergent on D, or i.e. it converges to f uniformly on D.

(Statement only)

Theorem 9.4.2 :- If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.

Proof :- $\boxed{\text{Thm 8.2.2 :- Let } \langle f_n \rangle \text{ be a seq}^n \text{ of continuous functions on a set } A \subseteq \mathbb{R} \text{ and suppose that } \langle f_n \rangle \text{ converges uniformly on } A \text{ to a function } f : A \to \mathbb{R}. \text{ Then } f \text{ is continuous on } A.}$

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(Statement only)

Theorem 9.4.3 :- Suppose that the real valued functions f_n, $n \in \mathbb{N}$, are Riemann integrable on the interval $J = [a,b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

(Statement only)

Theorem 9.4.4 :- For each $n \in \mathbb{N}$, let f_n be a real-valued function on $J = [a,b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum f_n'$ converges uniformly on J.

Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$

* Cauchy's Criteria for Uniform Convergence of Series of function *

Theorem 9.4.5 Let $\langle f_n \rangle$ be a seq^n of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D iff for every $\epsilon > 0$ there exist an $M(\epsilon)$ such that if $m > n \geq M(\epsilon)$, then

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \quad \forall \; x \in D.$$

Proof :-

Firstly, we will ~~state~~ state and prove cauchy criterion for uniform convergence, i.e. thm 8.1.10

Now we will prove the theorem.

let $\langle S_n \rangle$ be the seq^n of partial sums of the infinit series given by :-

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad ; \forall x \in D$$

By Cauchy Criterion for seq^n of functions, we have

$\langle S_n \rangle$ is uniformly convergent on D.

$\Longleftrightarrow$ for each $\epsilon > 0$, $\exists$ a natural no. $M(\epsilon)$ s.t.

$$|S_m(x) - S_n(x)| < \epsilon \quad \forall \; m > n \geq M(\epsilon)$$
$$\forall x \in D$$

$$\Leftrightarrow \left| \sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x) \right| < \epsilon \quad \forall \; m > n \geq M(\epsilon)$$
and $x \in D$.

$$\Leftrightarrow \quad \left| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x) \right| < \epsilon \qquad \forall\ m > n \geq M(\epsilon)\ \&\ \forall\ x \in D.$$

i.e. $\sum f_n$ is uniformly convergent on D iff for each $\epsilon > 0$, $\exists$ a natural no. $M(\epsilon)$ s.t.

$$\left| f_{n+1}(x) + \cdots + f_m(x) \right| < \epsilon \qquad \forall\ m > n \geq M(\epsilon),\ \forall\ x \in D$$

Hence proved

Theorem 9.4.6 :— "Weierstrass M-test"

Let $\langle M_n \rangle$ be a seq^n of +ve real numbers s.t. $|f_n(x)| \leq M_n$ $\forall\ x \in D$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof :— Let $\epsilon > 0$ be given arbitrary real numbers

For $m > n$, $m, n \in \mathbb{N}$,

Consider $\left| f_{n+1}(x) + \cdots + f_m(x) \right|$

$$\leq \left| f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x) \right|$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_m \quad \text{—①} \quad \left(\text{by using } |f_n(x)| \leq M_n\right.$$

$\forall\ x \in D.$

If $\sum M_n$ is convergent, then by Cauchy's criteria of series of real no., we get that

for given $\epsilon > 0$, $\exists$ a natural no. $K(\epsilon)$ s.t.

$$|M_{n+1} + M_{n+2} + \cdots + M_m| < \epsilon \quad \forall\ m > n \geq K(\epsilon)$$

from (1)

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| < \epsilon \quad \forall\ m > n \geq K(\epsilon) \quad \forall\ x \in D$$

$\therefore$ By Cauchy's criteria for series of functions, we get that $\sum f_n$ is uniformly convergent on D.

Hence proved

Exercises 9.4

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Q.1 Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by :-

[a] $(x^2+n^2)^{-1}$

Here, $$|f_n(x)| = \left| \frac{1}{x^2+n^2} \right| \quad \forall\ x \in \mathbb{R}$$

$$= \frac{1}{x^2+n^2} \quad \forall\ x \in \mathbb{R}$$

$$\leq \frac{1}{n^2} \quad \forall\ x \in \mathbb{R}$$

$\because x^2 \geq 0$
$n^2+x^2 \geq n^2$
$\Rightarrow \frac{1}{n^2+x^2} \leq \frac{1}{n^2}$

let $M_n = \frac{1}{n^2} \quad \forall n \in \mathbb{N}$

Then, $|f_n(x)| \le M_n \quad \forall n \in \mathbb{N}$ & $x \in \mathbb{R}$ —①

and $\sum M_n = \sum \frac{1}{n^2}$ is a convergent series by p-test.

So, by weierstrass M-test, it follows that the series

$\sum \frac{1}{x^2+n^2}$ is uniformly convergent on $\mathbb{R}$.

$\boxed{b}$ $(nx)^{-2}$; $(x \neq 0)$

Here $f_n(x) = \frac{1}{n^2 x^2}$; $x \neq 0$

Now, the series

$$\sum f_n = \sum \frac{1}{n^2 x^2} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is Convergent } \forall x \neq 0$$

(Since, the series $\sum \frac{1}{n^2}$ is cgt by p-test)

$\Rightarrow$ $\sum f_n$ is pointwise cgt $\forall$ $x \neq 0$ & $x \in \mathbb{R}$

Now $\left| \frac{1}{n^2 x^2} \right| \quad \forall x \neq 0, n \in \mathbb{N}$

$= \frac{1}{n^2 x^2} \quad , x \neq 0, n \in \mathbb{N}$

$\le \frac{1}{n^2 a^2}$ if $|x| \ge a$ for some $a > 0$

$\Rightarrow |f_n(x)| \le \frac{1}{n^2 a^2} \quad \forall x$ s.t. $|x| \ge a \quad \forall n \in \mathbb{N}$.

where $a>0$ is any real no.

$\Rightarrow \quad |f_n(x)| \le M_n \quad \forall\, x \in [a,\infty) \cup (-\infty,-a] \quad \forall\, n \in \mathbb{N}$

where $\quad M_n$ is $\dfrac{1}{n^2 a^2} \quad \forall\, n \in \mathbb{N}$

Now, $\sum M_n$ is a cgt series of real no.

$\therefore$ By W.M. test, $\sum f_n$ is uniformly convergent on the set $(-\infty,-a] \cup [a,\infty)$, where $a>0$ is arb. real no.

Now we will show that $\sum f_n$ is not uniformly cgt on $\mathbb{R}$. For this, it is sufficient to show that the seq^n $\langle f_n(x)\rangle$ is not uniformly cgt on $\mathbb{R}-\{0\}$

$\epsilon_0 = 1/2$

Let $\quad n_k = k \quad \& \quad x_k = \dfrac{1}{k} \in \mathbb{R}-\{0\} \quad \forall\, k \in \mathbb{N}$

$$|f_{n_k}(x_k)| = \left|\frac{1}{k^2 \times \frac{1}{k^2}}\right| = 1 > \epsilon_0 = \frac{1}{2} \quad \forall k \in \mathbb{N}$$

$\Rightarrow \quad \langle f_n \rangle$ is not uniformly cgt. on $\mathbb{R}-\{0\}$

$\Rightarrow \quad \sum f_n$ is not uniformly cgt on $\mathbb{R}-\{0\}$

$\boxed{c}$ $\quad \sin\left(\dfrac{x}{n^2}\right)$

Here $\quad f_n(x) = \sin\left(\dfrac{x}{n^2}\right) \quad \forall\, x \in \mathbb{R} \quad \forall\, n \in \mathbb{N}$

Now, $\quad |f_n(x)| = \left|\sin\left(\dfrac{x}{n^2}\right)\right| \quad \forall\, x \in \mathbb{R} \quad \& \quad \forall\, n \in \mathbb{N}$

$\qquad\qquad \le \left|\dfrac{x}{n^2}\right| \quad \forall\, x \in \mathbb{R}, \ \forall\, n \in \mathbb{N}$

$$= \frac{|x|}{n^2}$$

Let $M_n = \frac{|x|}{n^2}$

$\sum M_n = |x| \sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt by p-test

$\Rightarrow \sum f_n$ is absolutely cgt $\forall x \in \mathbb{R}$

$\Rightarrow \sum f_n$ is pointwise cgt on $\mathbb{R}$

Now,

$$|f_n(x)| \leq \frac{|x|}{n^2} \quad \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}$$

$$\leq \frac{a}{n^2} \quad \forall x \in \mathbb{R} \ \text{ s.t. } |x| \leq a \ (a>0)$$
$\forall n \in \mathbb{N}$

$\Rightarrow \sum f_n$ is uniformly cgt on $[-a, a]$, where $a>0$ is any real no.

Now, we will show that $\sum f_n$ is not uniformly cgt on $\mathbb{R}$

It is sufficient to show that $\langle f_n(x) \rangle$ is not uniformly cgt on $\mathbb{R}$.

Let $n_k = k$ and $x_k = \frac{\pi}{2} k^2$, $\epsilon_0 = 1/2$

then $\left| f_{n_k}(x_k) \right| = \left| \sin\left(\frac{\pi}{2} \cdot \frac{k^2}{k^2}\right) \right| = \left| \sin \frac{\pi}{2} \right| = 1 > \frac{1}{2} = \epsilon_0$

$\Rightarrow \langle f_n(x) \rangle$ is not uniformly cgt on $\mathbb{R}$

Hence, $\sum f_n(x)$ is not uniformly cgt on $\mathbb{R}$

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$\boxed{d}$ $(x^n+1)^{-1}$; $x \neq 0$

Here, $f_n(x) = \dfrac{1}{x^n+1}$ for $x>0$

Case-I $0 < x < 1$

Then $\lim_{n\to\infty} x^n = 0$

& $\lim_{n\to\infty} f_n(x) = \dfrac{1}{x^n+1} = \dfrac{1}{1+0} = 1 \neq 0$

$\Rightarrow$ $\sum f_n(x)$ is not cgt, for $x \in \mathbb{R}$ s.t. $|x|<1$

Case-II If $x=1$, then $f_n(x) = (1+1)^{-1} = \frac{1}{2}$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x) = \frac{1}{2} \neq 0$

$\Rightarrow$ $\sum f_n$ is not cgt at $x=1$

Case-III if $1<x<\infty$, then

$|f_n(x)| = \dfrac{1}{x^n+1}$ $\forall\ x>1$, $n \in \mathbb{N}$

$< \dfrac{1}{x^n}$ $\forall\ x>1$ $\forall\ n \in \mathbb{N}$

$= \left(\dfrac{1}{x}\right)^n$

$\Rightarrow$ $\sum \left(\frac{1}{x}\right)^n$ is a geometric series with common ratio $\frac{1}{x}$

$\Rightarrow$ $\sum \left(\frac{1}{x}\right)^n$ is cgt.

so, by comparison test $\sum |f_n(x)|$ is cgt

$\Rightarrow$ $\sum f_n(x)$ is cgt pointwise for $x>1$

Now, we will show that $\sum f_n$ is not uniformly cgt on $(1,\infty)$

let $n_k = k$ & $x_k = (2)^{1/k}$ $\in (1,\infty)$, $k \in \mathbb{N}$, $\epsilon_0 = 1/4$

then $|f_{n_k}(x_k)| = \left| \frac{1}{(2^{1/k})^k + 1} \right|$ $\forall k \in \mathbb{N}$

$= \frac{1}{2+1} = \frac{1}{3} > \frac{1}{4} = \epsilon_0$ $\forall k$

$\Rightarrow$ $\langle f_n(x) \rangle$ is not uniformly convergent on $(1,\infty)$

Hence

$\sum f_n$ is not uniformly cgt on $(1,\infty)$

$\boxed{e}$

$$f_n(x) = \frac{x^n}{x^n+1}, \quad x \geq 0$$

Case-I For $0 \leq x < 1$

we know $\frac{x^n}{x^n+1} \leq x^n$ $\forall x \geq 0$, $\forall n \in \mathbb{N}$

& $\sum (x)^n$ is a geometric series with common ration x & $0 \leq x < 1$

$\Rightarrow$ $\sum (x)^n$ is cgt

$\therefore$ By comparision test, $\sum f_n$ is cgt.

$x > 0$
$x^n > 0$
$1+x^n > 1$
$\frac{1}{1+x^n} < 1$
$\frac{x^n}{1+x^n} < x^n$

Case-II for $x=1$

$$f_n(x) = \frac{1}{2} \neq 0$$

$\Rightarrow$ $\sum f_n$ is not uniformly cgt. on $x=1$

Case-III $x > 1$

$$f_n(x) = \frac{1}{\left(\frac{1}{x}\right)^n + 1}$$

$\Rightarrow$ $\lim f_n(x) = 1 \neq 0$ $\Rightarrow$ $\sum f_n$ is not cgt for $x>1$

Hence, the series $\sum f_n$ is pointwise cgt on $[0,1)$

Now $|f_n(x)| = \dfrac{x^n}{x^n+1} \leq x^n \leq a^n$ for $0 \leq x \leq a$, $0 \leq a < 1$

Let $M_n = a^n$, $0 \leq a < 1$

then $|f_n(x)| \leq a^n$

& M_n is a geometric series

$\Rightarrow$ $\sum M_n$ is cgt

So by "W. M test" $\sum f_n$ is uniformly cgt on $[0,a]$ where $0 \leq a < 1$

Now, we will show that $\sum f_n$ is not uniformly cgt on $[0,1)$

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Let $n_k = k$ & $x_k = \left(\frac{1}{2}\right)^k \in [0,1)$

then

$$|f_{n_k}(x_k)| = \left| \frac{\left(\left(\frac{1}{2}\right)^{1/k}\right)^k}{\left(\left(\frac{1}{2}\right)^{1/k}\right)^k + 1} \right| \quad \forall k$$

$$= \left| \frac{\frac{1}{2}}{3/2} \right| = \frac{1}{3} > \frac{1}{4} = \epsilon_0$$

$\Rightarrow$ $\langle f_n \rangle$ is not uniformly cgt on ~~[0,1]~~ $[0,1)$

$\Rightarrow$ $\sum f_n$ is not uniformly cgt on $[0,1)$

—— x —— x ——