Lecture: 4 [Week 4 (05/04/2020 – 11/04/2020)]

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Section : A

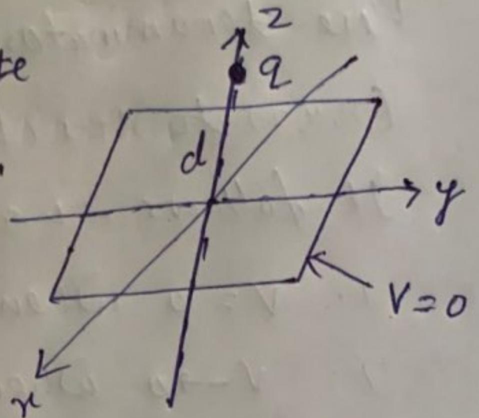
Paper Name : Electricity and Magnetism

Reference : David J. Griffiths, Introduction to Electrodynamics, 3rd Edt. 1999, Pearson Education Inc. Upper Saddle River, New Jersey 07458, U.S.A

Lecture - 4

The Method of Images

Suppose a charge q is held a distance d above infinite grounded conducting plane.



Q. What is the potential in the region above the plane?

Sol: Here the potential is not $\frac{1}{4\pi\epsilon_0}\frac{q}{r}$.

Here charge q will induce a certain amount of negative charge on the nearby surface of the conductor. Thus total potential is due to q and $q_{induced}$.

But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

For mathematical point of views is to solve Poisson equation in the region $z>0$ with a single point charge q at $(0,0,d)$ subject to the boundary condition.

1. $V=0$ when $z=0$ ($\because$ conducting plane is grounded)
2. $V\to 0$ far from the charge ($x^2+y^2+z^2 \gg d^2$)

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Here denominators represent the distance from (x,y,z) to the charges $+q$ and $-q$.

It follows that

1. $V=0$ when $z=0$
2. $V \to 0$ when $x^2+y^2+z^2 \gg d^2$

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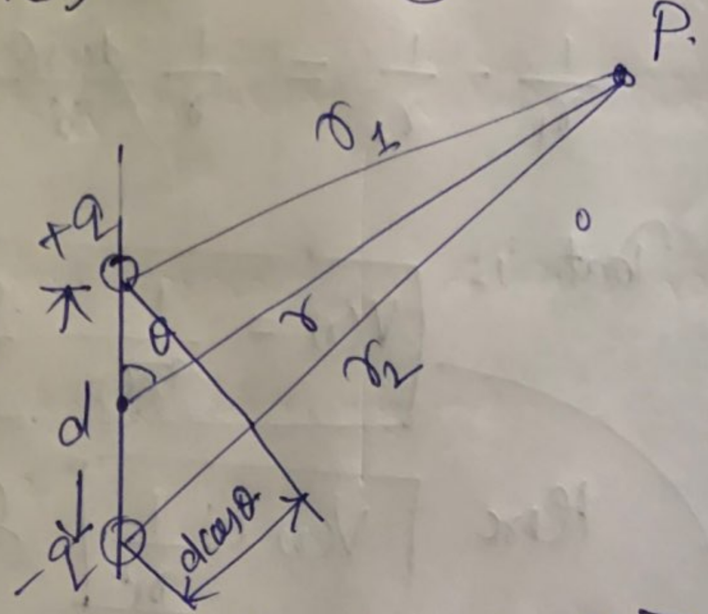
$V(r) = k\left[\frac{q}{r_1} - \frac{q}{r_2}\right]$ — (1)



If $r \gg d$ $\left[\begin{array}{l} r_1^2 = r^2 + \left(\frac{d}{2}\right)^2 - r d\cos\theta \\ r_2^2 = r^2 + (d/2)^2 + r d\cos\theta \end{array}\right]$

Then $r_2 - r_1 = d\cos\theta$, $r_2 r_1 = r^2$

In this case

$$r_1^2 = r^2\left[1 + \frac{d\cos\theta}{r} + \underbrace{\frac{d^2}{4r^2}}_{\to 0}\right]$$

$$r_2^2 = r^2\left[1 + \frac{d\cos\theta}{r} + \underbrace{\frac{d^2}{4r^2}}_{\to 0}\right]$$

for $r \gg d$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{d\cos\theta}{r}\right]^{-1/2}$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 + \frac{d\cos\theta}{2r}\right]$$

$$-\frac{1}{r_2} = -\frac{1}{r}\left[1 - \frac{d\cos\theta}{2r}\right]$$

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{r}\left[\cancel{1} + \frac{d\cos\theta}{2r} - \cancel{1} + \frac{d\cos\theta}{2r}\right]$$

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{r}\frac{d\cos\theta}{r} = \frac{d\cos\theta}{r^2}$$

That is $\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{q d\cos\theta}{r^2}}$

Here $\boxed{V(r) \propto \frac{1}{r^2}}$ for dipole.

We know that $d\cos\theta$

$$d\cos\theta = \vec{d}\cdot\hat{a}_r$$

and $\vec{d} = d\hat{a}_z$

$\vec{P} = q\vec{d}$ is the dipole moment.

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q\,\vec{d}\cdot\hat{a}_r}{r^2}$$

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}\cdot\hat{a}_r}{r^2}}$$

Here dipole moment $\vec{P}$ is directed from $-q$ to $+q$. If dipole center is not at the center but at r' then.

$$\boxed{V(r) = \frac{\vec{P}\cdot(\vec{r}-\vec{r}\,')}{4\pi\epsilon_0\,|r-r'|^3}}$$

The electric field due to the dipole with center at the origin.

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right]$$

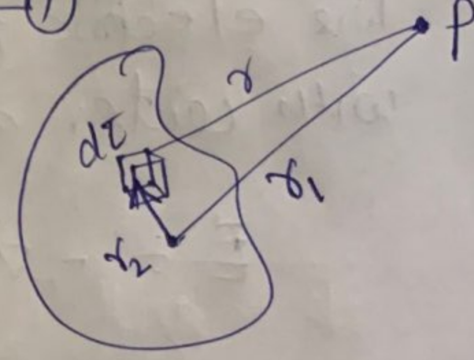
$$\vec{E} = -\frac{qd\cos\theta}{4\pi\epsilon_0}\times\frac{-2}{r^3}\hat{a}_r + \frac{qd\sin\theta}{4\pi\epsilon_0 r^2}\hat{a}_\theta$$

$$\vec{E} = \frac{qd}{4\pi\epsilon_0}\left[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right]$$

$$\boxed{\vec{E} = \frac{p}{4\pi\epsilon_0}\left[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right]}$$

Electric potential for continous charge distribution.

$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r)\,d\tau}{r} \quad -(1)$$



Using the law of cosines

$$r^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta$$

$$= r_1^2\left[1 + \left(\frac{r_2}{r_1}\right)^2 - 2\left(\frac{r_2}{r_1}\right)\cos\theta\right]$$

Let $r = r_1\sqrt{1+\epsilon} \quad -(2)$

where $\epsilon = \left(\frac{r_2}{r_1}\right)\left(\frac{r_2}{r_1} - 2\cos\theta\right)$

∴ for points well outside the charge distribution, ϵ is much small less than 1 then use Binomial Theorem.

$$\frac{1}{r} = \frac{1}{r_1}(1+\epsilon)^{-1/2}$$

$$= \frac{1}{r_1}\left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

(5)

$$\frac{1}{r} = \frac{1}{r_1}\left[1 - \frac{1}{2}\left(\frac{r_2}{r_1}\right)\left(\frac{r_2}{r_1} - \cos\theta\right) + \frac{3}{8}\left(\frac{r_2}{r_1}\right)^2\left(\frac{r_2}{r_1} - \cos\theta\right)^2 - \frac{5}{16}\left(\frac{r_2}{r_1}\right)^3\left(\frac{r_2}{r_1} - \cos\theta\right)^3 + \dots\right].$$

$$\cancel{= \frac{1}{r_1}\left[1 - \frac{1}{2}\left(\frac{r_2}{r_1}\right)^2 + \frac{1}{2}\left(\frac{r_2}{r_1}\right)\cos\theta + \frac{3}{8}\left(\frac{r_2}{r_1}\right)^3\right.}$$

$$= \frac{1}{r_1}\left[1 + \left(\frac{r_2}{r_1}\right)\cos\theta + \left(\frac{r_2}{r_1}\right)^2 \frac{(3\cos^2\theta - 1)}{2} + \left(\frac{r_2}{r_1}\right)^3 \left(\frac{5\cos^3 - 3\cos\theta}{2}\right) + \dots\right]$$

$$\frac{1}{r} = \frac{1}{r_1}\sum_{n=0}^{\infty}\left(\frac{r_2}{r_1}\right)^n P_n(\cos\theta)$$

Thus

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r_1^{n+1}}\int r_2^n P_n(\cos\theta)\,\rho(r)\,d\tau}$$

Or

$$V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r_1}\int \rho(r)\,d\tau + \frac{1}{r_1^2}\int r_2\cos\theta\,\rho(r)\,d\tau + \frac{1}{r_1^3}\int r_2^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho(r)\,d\tau + \dots\right]$$

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Q.

A sphere of radius R centered at the origin, carries a density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

Where k is a constant, and r, θ are usual spherical coordinates. find the approximate potential for points on the z-axis far from the ~~too~~ sphere.

H.W

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* The monopole and Dipole Terms

$$\boxed{V_{mon}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}}$$

$Q = \int \rho \, d\tau$ is the total charge.

$$V_{dip}(r) = \frac{1}{4\pi\epsilon_0}\frac{1}{r_1^2}\int r_2 \cos\theta \, \rho(r) \, d\tau$$

Since θ is the angle b/w $\vec{r}_2$ and $\vec{r}$.

$$\hat{r}\cdot\vec{r}_2 = |\hat{r}||r_2|\cos\theta$$

$$\hat{r}\cdot\vec{r}_2 = r_2\cos\theta$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r_1^2}\int \hat{r}\cdot\vec{r}_2 \, \rho(r) \, d\tau$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r_1^2}\hat{r}\underbrace{\int \vec{r}_2 \, \rho(r) \, d\tau}_{\vec{p}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r_1^2}\hat{r}\cdot\vec{p}$$

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Thus

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r_1^2}$$

Or Generally

$$\boxed{V_{dip}(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}}$$

where $\vec{p} = \sum q_i r_i$

$$P = q(r_2 - r_1)$$

$$\boxed{p = q d}$$

t
$= dt$
$\frac{4\pi k R^3}{3}$