(S&H) **Difference Equations : chap 20·1**

In a dynamic framework, the values of economic variables change (steadily or with oscillations) with time. The laws governing the behaviour of these variables are usually expressed in terms of one or more equations.

→ If time is taken to be a discrete (or, integer valued) variable and the equations relate the values of such variables at different points of time, then we are confronted with difference equations or recurrence relations.

→ In this case time is usually measured by simply counting forward the number of periods that have elapsed after an initial time $t = 0$.

→ → Sometimes, however we also consider negative times, in which case $t = 0$ should be thought of as the origin of our time measure.

→ If time is regarded as a continuous variable and the equations involve unknown functions and their derivatives → differential equations

(see over...)

20.1 First Order Difference Equations:

→ Changes in many of the quantities economists study (such as income, consumption and savings) are usually observed at fixed time intervals (eg, each day, week or year).

→ These quantities are then dated according to the period to which they refer and the behaviour of these economic variables is studied at discrete moments of time.

(*) → Equations that relate such quantities at different times are called <u>difference equations</u>. eg, such an equation might relate the amount of national income in one period to the national income in one or more previous periods.

(*) → Let $f(t, x)$ be a fun defined for all positive integers t and all real numbers x. A fairly general difference equation of the <u>first order</u> is:

$$x_t = f(t, x_{t-1}) \; ; \; t = 1, 2, \ldots \quad (1)$$

< (*) The general difference equation

of the first order is $F(t, x_{t-1}, x_t) = 0$. If this equation can be solved for x_t in terms of t and x_{t-1} we have equation ①

④ → This is a first order equation because, it relates the value of a fun in each period t to the value of the same fun in the previous period $(t-1)$ only. ⟨ when we deal with fun defined for discrete time periods, we usually let x_t [rather that $x(t)$] denote the value of the variable at time t ⟩

④ Note : Some people refer to equations of type ① as recurrence relations and would insist that a difference equation is one where the difference $\Delta x_t = x_t - x_{t-1}$ is specified as a fun of t and x_{t-1}. But equation ① is evidently equivalent to the difference equation $\Delta x_t = f(t, x_{t-1}) - x_{t-1}$

→ Conversely, given a difference equation $\Delta x_t = g(t, x_{t-1})$ there is a corresponding recurrence relation $x_t = x_{t-1} + g(t, x_{t-1})$

(see over ...)

→ Because of the obvious correspondence b/w recurrence relations and difference equations, there seems no good reason to maintain a distinction b/w the two.

→ So, we will refer ① as a difference equation.

→ Suppose x_0 is given.

Then from successive insertions of time periods in ① yields

$$x_1 = f(1, x_0)$$

$$x_2 = f(2, x_1) = f[2, f(1, x_0)]$$

$$x_3 = f(3, x_2) = f[3, f\{2, f(1, x_0)\}]$$

and so on.

→ For a given value of x_0 we can compute x_t for any value of t.

→ Formally we state this simple result as :

Theorem 1 : [Existence and Uniqueness theorem] Consider the difference equation $x_t = f(t, x_{t-1})$; $t = 1, 2, \ldots$ where f is defined for all values of the variables. If x_0 is an arbitrary fixed number then there exists a uniquely determined fun x_t that is a soln of the equation and has a given value for $t = 0$

→ In general for each choice of x_0, there is a different corresponding unique soln of ①. Consequently, there are infinitely many solns.

→ The existence and uniqueness theorem for ① is almost trivial. It implies that when x_0 is given, the successive values of x_t can be computed for any natural number t.

[④ Actually, we need to know more, which we take up gradually in a very restrictive manner].

➤ First-Order Equations with a Constant Coefficient:

We study first the linear difference equation:

$$x_t = a x_{t-1} + b_t \; ; \; (t = 1, 2, \ldots) \qquad \text{②}$$

Starting with a given x_0, it is possible to calculate x_t algebraically for small t.

(see over...)

$x_1 = a x_0 + b_1$

$x_2 = a x_1 + b_2 = a(a x_0 + b_1) + b_2$

$= a^2 x_0 + a b_1 + b_2$

$x_3 = a x_2 + b_3 = a(a^2 x_0 + a b_1 + b_2) + b_3$

$= a^3 x_0 + a^2 b_1 + a b_2 + b_3$

- - - - - - - - - and so on - - - - -

$\rightarrow$ This makes the pattern clear.

Then —

$\boxed{\begin{array}{l} \text{The difference equation} \\ x_t = a x_{t-1} + b_t \;;\; t = 1, 2, \ldots \\ \text{has the soln} \qquad \text{——— (3)} \\ x_t = a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k \end{array}}$

(*) **Note** : $a^{t-k} = a^0 = 1$ when $t = k$

(*) $\rightarrow$ To check that (3) is really a soln to (2), substituting the expression in (3) for x_{t-1} into the RHS of (2) yields ;

$$a x_{t-1} + b_t$$

$$= a\left[a^{t-1} x_0 + \sum_{k=1}^{t-1} a^{t-1-k} b_k\right] + b_t$$

$$= a^t x_0 + \sum_{k=1}^{t-1} a^{t-k} b_k + b_t$$

$$= a^t x_0 + \sum_{k=1}^{t} a^{t-k} b_k.$$

→ This reaches our expression for x_t in (3).

So, (3) does solve the difference equation

(*) Consider, the special case when $b_k = b \;\forall\; k = 1, 2, \ldots$

then, $$\sum_{k=1}^{t} a^{t-k} \cdot b_k = b \sum_{k=1}^{t} a^{t-k}$$

$$= b\,[\underbrace{a^{t-1} + a^{t-2} + \cdots + a + 1}_{\text{standard geometric series.}}]$$

$$= b \cdot \frac{1-a^t}{1-a} \quad \text{for } a \neq 1$$

(see overleaf)

(*) Thus for $t = 1, 2, \ldots$

$$\boxed{x_t = a x_{t-1} + b \iff x_t = a^t\left[x_0 - \frac{b}{1-a}\right] + \frac{b}{1-a} \quad \text{for } a \neq 1} \quad \text{--- (4)}$$

Here, (for $a \neq 1$)

$$x_t = a^t x_0 + b \cdot \frac{1-a^t}{1-a}$$

$$= a^t x_0 + \frac{b}{1-a} - \frac{b a^t}{1-a}$$

$$= a^t\left[x_0 - \frac{b}{1-a}\right] + \frac{b}{1-a}$$

(*) $\boxed{\text{For, } a = 1}$

$$1 + a + a^2 + \cdots a^{t-1} = t$$

and hence, $\boxed{x_t = x_0 + bt}$

for $t = 1, 2, \ldots$

since, $a^t x_0 = (1)^t x_0 = x_0$

(*) See examples from the book.

Equilibrium States and Stability :

→ Consider the soln of $x_t = ax_{t-1} + b$ given in (4), If $x_0 = \frac{b}{1-a}$, then

$$x_t = a^t \underbrace{\left[\frac{b}{1-a} - \frac{b}{1-a}\right]}_{= \text{zero}} + \frac{b}{1-a} = \frac{b}{1-a} \ \forall t.$$

In fact, if $x_s = \frac{b}{1-a}$ for some $s \geq 0$, then

$$x_{s+1} = ax_s + b = a\frac{b}{1-a} + b = \frac{b}{1-a}$$

Also, $x_{s+2} = ax_{s+1} + b = a \cdot \frac{b}{1-a} + b = \frac{b}{1-a}$.

and so on.

→ We conclude that if x_s ever becomes equal to $\frac{b}{1-a}$ at some time s then x_t will remain at this constant level for each $t \geq s$.

→ The constant,

$$x^* = \frac{b}{1-a} \quad \text{———} \quad (5)$$

is called an equilibrium (or, stationary) state for $x_t = ax_{t-1} + b$, when $a \neq 1$.

(see over...)

(*) **Note**: An alternative way of finding x^* is to seek a solution of $x_t = a x_{t-1} + b$ with, $x_t = x^* \; \forall t$.

→ Such a soln must satisfy.

$x_t = x_{t-1} = x^*$, hence, $x^* = ax^* + b$

$\Rightarrow x^* = \dfrac{b}{1-a}$ for $(a \neq 1)$.

→ concept, of intertemporal equm

→ Defn (5) allows us to rewrite (4) as:

$$\boxed{x_t - x^* = a(x_{t-1} - x^*) \quad (6) \\ \Longleftrightarrow x_t - x^* = a^t(x_0 - x^*)}$$

(*) **Note** that $(x_t - x^*)$ is the deviation of x_t from its equm value. So, (6) shows that this deviation grows (or, shrinks) at the constant proportional rate $(a-1)$

→ Suppose the constant a is less than one (1) in absolute value — that is $|a| < 1$ or, $-1 < a < 1$, then $a^t \to 0$ as $t \to \infty$

→ So, (4) implies that $x_t \to x^* = \frac{b}{1-a}$ as $t \to \infty$

→ Hence, if $|a| < 1$, the soln in (4) converges to the equm state when $t \to \infty$. The equation is then called stable

→ Two types of convergence, (i) monotonic (or, gradual) and (ii) damped oscillations (Ref fig 1)

→ If $|a| > 1$, then the ~~asto~~ absolute value of a^t approaches ∞ as $t \to \infty$. From (4) it follows that x_t moves further and further away from the equm state, except when $x_0 = \frac{b}{1-a}$.

→ The soln in (4) diverges from the equm state as $t \to \infty$.

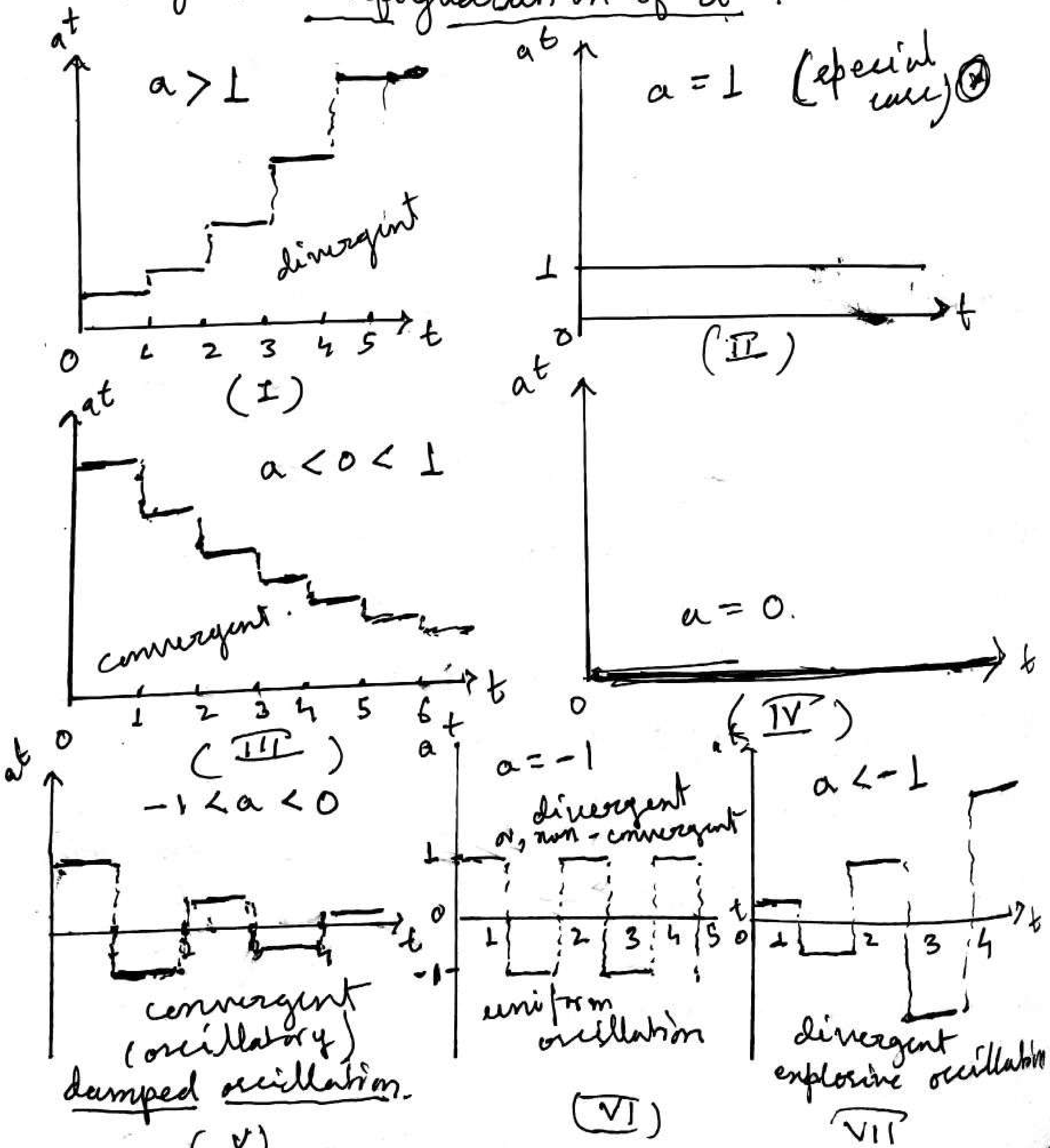
→ Two types of divergence, (i) monotonic (or, gradual) and (ii) explosive oscillations. (Ref fig 1).

(see over....)

→ If $a = -1$ then we have uniform oscillations → which is a case for divergence once again

→ If $a = 1$ ⟨different soln⟩ we call this a case of moving eqm.

Fig 1: Configuration of a^t.



(*) → Unlike the book's diagrams, the diagrams would be step fun s, since t is integer valued (discrete).

(*) Crux of the matter :

→ Dynamic analysis with discrete time.
→ according to the syllabus difference is of one time period.
→ if it is like. $x_{t+2} = ax_{t+1} + b$
then it is equivalent to $x_t = ax_{t-1} + b$.

→ Hence, given an equation, first you have to standardise it into the given form and then use formula according to the value of 'a'. <Note : since, the derivation of the formula is not that difficult it could be asked as a theory question also>

→ Given you have a soln (time path), you can check stability (convergence towards the intertemporal eqlm / stationary state) according to the value of 'a'. <provide diagram of the time path if only mentioned specifically>.

(*) See the examples in the book and try the exercises also.

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