B.Sc. Prog. Computer Science Semester VI^{th}
Numerical method - By Kapil Kumar
Chapter-2 Tuesday 24/03/2020 Lec-1,2

Ex:-11.) Perform two iterations of Newton's method to solve the non-linear system of equations with initial approximation (1,1).

$$f(x,y) = x^2 + y - 11 = 0 \;\&$$
$$g(x,y) = x^2 + y^2 - 7 = 0$$

Soln:- $f_x = 2x$, $f_y = 1$, $g_x = \cancel{2x}\ 1$, $g_y = 2y$

We have $f_x(x_0,y_0)\, r + f_y(x_0,y_0)\, s = -f(x_0,y_0)$

$g_x(x_0,y_0)\, r + g_y(x_0,y_0)\, s = -g(x_0,y_0)$

Substituting in the formula we get (I)

$$2r + s = 9$$
$$\cancel{2}r + 2s = 5$$

Solving these equation we get

$$r = \frac{13}{3} \;\&\; s = \frac{1}{3}$$

Next approximation is $x_1 = x_0 + r = 1 + \frac{13}{3} = \frac{16}{3} = 5.33$

$$y_1 = y_0 + s = 1 + \frac{1}{3} = \frac{4}{3} = 1.33$$

Now we have $f_x(x_1,y_1)\, r + f_y(x_1,y_1)\, s = -f(x_1,y_1)$

$g_x(x_1,y_1)\, r + g_y(x_1,y_1)\, s = -g(x_1,y_1)$

$$2\left(\frac{16}{3}\right) r + s = 11 - \left(\frac{16}{3}\right)^2 - \frac{4}{3}$$

$$r + 2 \times \frac{4}{3} s = 7 - \left(\frac{16}{3}\right) - \left(\frac{4}{3}\right)^2$$

Scanned with CamScanner

$$10.66\,r + s = -11.7389$$
$$r + 2.66\,s = -0.0989$$

On solving we get

$$r = -1.10143,\quad s = 0.37689$$

$$x_2 = x_1 + r = 5.33 - 1.10143 = 4.22857$$
$$y_2 = y_1 + s = 1.33 + 0.37689 = 1.70689$$

(b) Indirect method (Matrix method):

For this let us first consider a system of two non-linear equations as $f(x,y)=0$ & $g(x,y)=0$

Consider (x_n, y_n) be in approximation to the root (ξ, η) of system.

First we find, $J_n = \begin{bmatrix} f_x(x_n,y_n) & f_y(x_n,y_n) \\ g_x(x_n,y_n) & g_y(x_n,y_n) \end{bmatrix}$

$$J_n^{-1} = \frac{1}{D_n} \cdot J_n^{T}, \quad D_n = |J_n|$$

Then we can write new system

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{1}{D_n} J_n^{T} F(x_n, y_n) \qquad (1)$$

$n = 0, 1, 2$ ——

where $F(x_n, y_n) = \begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix}$

Using initial approximation (x_0, y_0) we can find out next approximation for $n = 0, 1, 2$ ——

Ex:(1.) Perform three iteration of Newton Raphson method to solve the system of non-linear equation

$$x^2+xy+y^2=19 \quad \text{with initial approximation}$$
$$x^3+y^3=35 \quad (x_0,y_0)=(1.5,1)$$

Soln:-
$$f(x,y)=x^2+xy+y^2-19=0$$
$$g(x,y)=x^3+y^3-35=0$$

$$J_n=\begin{bmatrix} f_x(x_n,y_n) & f_y(x_n,y_n) \\ g_x(x_n,y_n) & g_y(x_n,y_n) \end{bmatrix}=\begin{bmatrix} 2x_n+y_n & x_n+2y_n \\ 3x_n^2 & 3y_n^2 \end{bmatrix}$$

$$J_n^{-1}=\frac{1}{D_n}J_n^T=\frac{1}{D_n}\begin{bmatrix} 3y_n^2 & -(x_n+2y_n) \\ -3x_n^2 & 2x_n+y_n \end{bmatrix}$$

$$D_n=|J_n|=3y_n^2(2x_n+y_n)+3x_n^2(x_n+2y_n)$$

$$F(x_n,y_n)=\begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix}=\begin{bmatrix} x_n^2+x_ny_n+y_n^2-19 \\ x_n^3+y_n^3-35 \end{bmatrix}$$

Now we can write

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}=\begin{bmatrix} x_n \\ y_n \end{bmatrix}-\frac{1}{D_n}\begin{bmatrix} 3y_n^2 & -(x_n+2y_n) \\ -3x_n^2 & 2x_n+y_n \end{bmatrix}\begin{bmatrix} x_n^2+x_ny_n+y_n^2-19 \\ x_n^3+y_n^3-35 \end{bmatrix} \quad (1)$$

For Ist interation put $n=0$ & $(x_0,y_0)=(1.5,1)$ in eq^n (1)

$$\begin{bmatrix} x_{1} \\ y_{1} \end{bmatrix}=\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}-\frac{1}{D_0}\begin{bmatrix} 3y_0^2 & -(x_0+2y_0) \\ -3x_0^2 & 2x_0+y_0 \end{bmatrix}\begin{bmatrix} x_0^2+x_0y_0+y_0^2-19 \\ x_0^3+y_0^3-35 \end{bmatrix}$$

$$D_0=3y_0^2(2x_0+y_0)+3x_0^2(x_0+2y_0)$$
$$=-11.625$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} + \frac{1}{11.62}\begin{bmatrix} 3(1)^2 & -(1.5+2) \\ -3(1.5)^2 & 3+1 \end{bmatrix}\begin{bmatrix} (1.5)^2+1.5+1-19 \\ (1.5)^3+(1)^3-35 \end{bmatrix}$$

$$= \begin{bmatrix} 7.0454 \\ -1.2644 \end{bmatrix}$$

Similarly we will find out 2nd Iteration Put $n=1$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \frac{1}{D_1}\begin{bmatrix} 3y_1^2 & -(x_1+2y_1) \\ -3x_1^2 & 2x_1+y_1 \end{bmatrix}\begin{bmatrix} x_1^2+x_1y_1+y_1^2-19 \\ x_1^3+y_1^3-35 \end{bmatrix}$$

$$D_1 = -611.0634$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7.0454 \\ -1.2644 \end{bmatrix} + \frac{1}{611.0634}\begin{bmatrix} 4.796 & -4.5166 \\ -148.913 & 12.8264 \end{bmatrix}\begin{bmatrix} 23.3282 \\ 312.696 \end{bmatrix} =$$

$$= \begin{bmatrix} 4.91724 \\ -0.38578 \end{bmatrix}.$$

Ex:-(2.) Perform two Iterations of Newton's method to solve the non-linear system of equations with initial approximation $(1,1)$

$$f(x,y) = x^2+y^2-4 = 0 \ \&$$
$$g(x,y) = x^2+y^2-16 = 0$$

Soln:- $J_n = \begin{bmatrix} f_x(x_n,y_n) & f_y(x_n,y_n) \\ g_x(x_n,y_n) & g_y(x_n,y_n) \end{bmatrix} = \begin{bmatrix} 2x_n & 2y_n \\ 2x_n & 2y_n \end{bmatrix}$

$$J_n^{-1} = \frac{1}{D_n} J_n^T = \frac{1}{D_n}\begin{bmatrix} 2y_n & -2y_n \\ -2x_n & 2x_n \end{bmatrix}$$

$$D_n = |J_n| = 4x_ny_n - 4x_ny_n = 0$$

so unique solution not exist.

Rate of convergence & Termination Point :-

Rate of convergence :- An iterative method is said to be of order p or has the rate of convergence p, if p is the largest ~~possible~~ positive real number for which there exist a finite constant $c \neq 0$ such that

$$|\epsilon_{k+1}| \leq c\,|\epsilon_k|^p$$ where $\epsilon_k = x_k - \xi_e$ is the error in the kth iterate. The constant c, which is independent of k, is called the asymptotic error constant and it is depends on the derivative of $f(x)$ at $x = \xi_e$.

Termination point :- In order to reach the end of the process we have some idea of accuracy depending on what answer we need. So for this, we consider pre-determined tolerance E and stop the sequence when the error is smaller than the tolerance, that is $|\epsilon_n| < E$. This is called stopping condition or Termination Point.

Rate of convergence of False Position Method (Regula Falsi Method) :-

We have noted earlier that if the root lies initially in the interval (x_0, x_1), then one of the end points is fixed for all iterations. If the left end point x_0 is fixed and the right end point moves towards the required root. Using the formula :- $$x_{k+1} = \frac{x_0 f_k - x_k f_0}{f_k - f_0}$$

Substituting $x_k = \epsilon_k + \xi_e$, $x_{k+1} = \epsilon_{k+1} + \xi_e$ & $x_0 = \epsilon_0 + \xi_e$, we expand each term in Taylor's series and simplify using the fact that $f(\xi)=0$ 12.

$$\epsilon_{k+1}+\xi = \frac{(\epsilon_0+\xi)f(\epsilon_k+\xi)-(\epsilon_k+\xi)f(\epsilon_0+\xi)}{f(\epsilon_k+\xi)-f(\epsilon_0+\xi)}$$

we obtain the error equation as

$$\epsilon_{k+1} = C\epsilon_0\epsilon_k \quad \text{where } C=\frac{f''(\xi)}{2f'(\xi)}$$

Since ϵ_0 is finite and fixed, the error equation becomes

$$|\epsilon_{k+1}| = |C^*||\epsilon_k| \quad \text{where } C^*=C\epsilon_0$$

Hence, the method of false position has order 1 or has linear rate of convergence.

Rate of convergence of Secant method:- we assume that ξ is a simple root of $f(x)=0$ i.e. $f(\xi)=0$

Substituting $x_k=\xi+\epsilon_k$ in the formula $x_{k+1}=\epsilon_{k+1}+\xi$

$$x_{k+1} = x_k - \frac{x_k-x_{k-1}}{f_{x_k}-f_{x_{k-1}}}f_{x_k}, \quad k=1,2 -$$

$$\epsilon_{k+1}+\xi = \epsilon_k+\xi - \frac{(\epsilon_k+\xi)-(\epsilon_{k-1}+\xi)}{f(\epsilon_k+\xi)-f(\epsilon_{k-1}+\xi)}f(\epsilon_k+\xi)$$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k-\epsilon_{k-1})f(\epsilon_k+\xi)}{f(\xi+\epsilon_k)-f(\xi+\epsilon_{k-1})}$$

Expanding $f(\xi+\epsilon_k)$ & $f(\xi+\epsilon_{k-1})$ in Taylor series about the point ξ and putting $f(\xi)=0$, we get.

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k-\epsilon_{k-1})\left[\epsilon_k f'(\xi)+\frac{1}{2}\epsilon_k^2 f''(\xi)+ \cdots\right]}{(\epsilon_k-\epsilon_{k-1})f'(\xi)+\frac{1}{2}(\epsilon_k^2-\epsilon_{k-1}^2)f''(\xi)+ \cdots}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k f'(\xi) + \frac{1}{2}\epsilon_k^2 f''(\xi) + \cdots\right]}{(\epsilon_k - \epsilon_{k-1})\left[f'(\xi) + \frac{1}{2}(\epsilon_k + \epsilon_{k-1}) f''(\xi) + \cdots\right]}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{1}{2}\epsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{1}{2}(\epsilon_k + \epsilon_{k-1})\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\epsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\epsilon_k\epsilon_{k-1} + O(\epsilon_k^2\epsilon_{k-1} + \epsilon_k\epsilon_{k-1}^2)$$

$\epsilon_{k+1} = C\epsilon_k\epsilon_{k-1}$ (1) where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and higher power of ϵ_k are neglected.

eqn (1) is called error equation. Now from definition of rate of convergence, we seek a relation of the form

$\epsilon_{k+1} = A\epsilon_k^p$ we have to determine A & p

$$\epsilon_k = A\epsilon_{k-1}^p \quad \text{or} \quad \epsilon_{k-1} = \epsilon_k^{1/p} A^{-1/p}$$

Substituting the value of ϵ_{k+1} & ϵ_{k-1} in error eqn

$$A\epsilon_k^p = C\epsilon_k \epsilon_k^{1/p} A^{-1/p}$$

$$\epsilon_k^p = CA^{-(1+\frac{1}{p})}\epsilon_k^{1+\frac{1}{p}}$$

Comparing the power of ϵ_k on both sides, we get

$$p = 1 + \frac{1}{p} \Rightarrow p^2 - p + 1 = 0 \quad p = \frac{1}{2}(1 \pm \sqrt{5})$$

Neglecting the minus sign, we find the rate of convergence for secant method is $p = 1.618$.

<u>Rate of convergence of Newton-Raphson method</u> :- We assume that ξ is a simple root of $f(x) = 0$ i.e $f(\xi) = 0$ Substituting $x_k = \xi + \epsilon_k$ & $x_{k+1} = \xi + \epsilon_{k+1}$

in the formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$$\epsilon_{k+1} + \xi = \epsilon_k + \xi - \frac{f(\xi+\epsilon_k)}{f'(\xi+\epsilon_k)}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{f(\xi+\epsilon_k)}{f'(\xi+\epsilon_k)}$$

Expanding $f(\xi+\epsilon_k)$ & $f'(\xi+\epsilon_k)$ in Taylor series about the point ξ & putting $f(\xi)=0$, we get

$$\epsilon_{k+1} = \epsilon_k - \frac{\left[\epsilon_k f'(\xi) + \frac{1}{2}\epsilon_k^2 f''(\xi) + \cdots\right]}{f'(\xi) + \epsilon_k f''(\xi) + \cdots}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{f''(\xi)}{2f'(\xi)}\epsilon_k^2 + \cdots\right]\left[1 + \frac{f''(\xi)}{f'(\xi)}\epsilon_k + \cdots\right]^{-1}$$

$$= \epsilon_k - \left[\epsilon_k + \frac{f''(\xi)}{2f'(\xi)}\epsilon_k^2 + \cdots\right]\left[1 - \frac{f''(\xi)}{f'(\xi)}\epsilon_k + \cdots\right]$$

$$= \frac{f''(\xi)}{2f'(\xi)}\epsilon_k^2 + O(\epsilon_k^3 + \cdots)$$

$\epsilon_{k+1} = c\,\epsilon_k^2$ (3) where $c = \frac{f''(\xi)}{2f'(\xi)}$ and higher power of ϵ_k are neglecting. E_ξ^n (3) is error equation. Now from definition of rate of convergence

$$|\epsilon_{k+1}| = |c|\,|\epsilon_k|^2$$

Therefore, Newton's method is of order 2 or has quadratic rate of convergence.