CHAPTER-7 Method of Separation of Variables — ①

In the previous chapter(5) we have discussed. d' Alembert Solution of the Cauchy problem for one dimensional wave eq. Now, we shall discuss the method of separation of variables, for solving I.B.V.P. This method is applicable for a wide range of problems of mathematical physics. This method is simplier and easy to apply as it transform the given 2^{nd} order P.D.E into a set of O.D.E's which can be easily integrated to find the solution.

Q.1. Consider a homogenous rod of length l. The rod is sufficiently thin so that the heat is distributed equally over the cross section at time t. The surface of the rod is insulated, & therefore, there is no heat loss through the boundary. The temp. distribution of the rod is given by the solution of the initial boundary value problem.

Consider a homogeneous rod of length l. The rod is sufficiently thin so that the heat is distributed equally over the cross section at time t.

1) Heat Conduction Problem

$$u_t = k u_{xx}, \quad 0<x<l, \ t>0$$
$$u(0,t) = 0, \quad t\geq 0$$
$$u(l,t) = 0, \quad t\geq 0$$
$$u(x,0) = f(x), \quad 0\leq x\leq l$$

Let us assume the solution in the form

$$u(x,t) = X(x)\,T(t) \neq 0$$

$$\Rightarrow u_t = X(x)\,T'(t), \text{ and } u_{xx} = X''(x)\,T(t)$$

Sub. in the given eq^n, we get

$$XT' = kX''T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} = -\alpha^2 \text{ (say)}, \text{ where } \alpha>0 \qquad \text{— (1)}$$

$\Rightarrow$ X and T must satisfy

$$X''+\alpha^2 X = 0 \quad \text{and} \quad T'+\alpha^2 kT = 0$$

From the boundary conditions, we have

$u(0,t) = X(0)\,T(t) = 0$ and $u(l,t) = X(l)\,T(t) = 0 \quad \forall\, t>0$

$\Rightarrow X(0)=0$ and $X(l)=0$ for and arbitrary f^n $T(t)$

Now we solve the eigen value problem

$$X''+\alpha^2 X = 0 \ ; \ X(0)=0, \ X(l)=0$$

$\Rightarrow X(x) = A\cos\alpha x + B\sin\alpha x$, using $X(0)=0$ & $X(l)=0$

we get $A=0$ and $B\sin\alpha l = X(l) = 0$

If $B=0$, we get a trivial solution $\Rightarrow B\neq 0$

$\therefore \sin\alpha l = 0 \Rightarrow \alpha l = n\pi \Rightarrow \alpha = \frac{n\pi}{l}$, for $n=1,2,\ldots$

$\therefore$ we get $\quad X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \qquad \text{— (4)}$

Now consider $\quad T'+\alpha^2 kT = 0 \Rightarrow T = Ce^{-\alpha^2 kt}$

Putting $\alpha = \frac{n\pi}{l}$, we have $\quad T_n(t) = C_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \qquad (5)$

$\therefore$ the non-trivial solution of the heat eq^n which satisfies the two boundary conditions is

$$u_n(x,t) = X_n(x)\,T_n(t) = a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l} \ ; \ n=1,2,\ldots$$
$$a_n = B_n C_n.$$

By the principle of superposition, we obtain the series solution as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$
$$= \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$ which satisfy the

initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

This holds true if $f(x)$ can be represented by fourier sine series with fourier co-efficients

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Hence $$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx\right] e^{-\left(\frac{n\pi}{l}\right)^2 k t} \sin\frac{n\pi x}{l}} \quad \text{—} (7)$$

Q2 Now consider a problem of vibrating string of constant tension T^* & density ρ with $c^2 = T^*/\rho$ stretched along the x-axis from 0 to l, fixed at its end points.

The problem is given by.

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0<x<l, \quad t>0$$

$$\left.\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \end{aligned}\right\} \quad 0 \leq x \leq l$$

$$u(0,t) = u(l,t) = 0 \qquad , t \geq 0.$$

Here f & g are initial displacement & initial vel. resp.

Sol. By the method of sep. of variables, we assume a sol. in the form

$$u(x,t) = X(x)\,T(t) \neq 0.$$

Proceeding as in the case of Heat Conduction Problem we found the solution of Wave equation can be expressed in the form of an infinite series.

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}$$

~~where~~ and applying initial conditions we get

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l}$$

These eq's will be satisfied if $f(x)$ & $g(x)$ can be represented by Fourier sine series. The coefficients are given by

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad , \quad b_n = \frac{2}{n\pi c}\int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

EX 7.9 Pg 265 Using method of Separation of Variables

Q1 a) Solve the following I.B.V.P.

a) $u_{tt} = c^2 u_{xx} \quad 0<x<1 \ , \ t>0$

$u(x,0) = x(1-x) \ , \ u_t(x,0) = 0, \quad 0 \le x \le 1$

$u(0,t) = u(1,t) = 0, \quad t>0.$

Sol. We know that sol. of vibrating string prob. (Wave eq.) is given by.

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c}{l} t + b_n \sin \frac{n\pi c}{l} t \right) \sin \frac{n\pi x}{l} \quad \text{—— (1)}.$$

where $a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \ , \ b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx.$

$\therefore \quad a_n = \frac{2}{1} \int_0^1 x(1-x) \sin n\pi x \, dx.$

$= 2\{\int_0^1 x \sin n\pi x \, dx - \int_0^1 x^2 \sin n\pi x \, dx.$

Integ. by parts.

$= 2\left[-x \frac{\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi} dx + x^2 \frac{\cos n\pi x}{n\pi}\Big|_0^1 - \int_0^1 2x \frac{\cos n\pi x}{n\pi} dx \right].$

$= -\frac{2(-1)^n}{n\pi} + 0 + \frac{\sin n\pi x}{(n\pi)^2}\Big|_0^1 + \frac{2(-1)^n}{n\pi} - 0 - \frac{4}{n\pi}\left[x \frac{\sin n\pi x}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi} dx \right]$

$= -\frac{4}{(n\pi)^2} \frac{\cos n\pi x}{n\pi}\Big|_0^1 = \frac{-4}{(n\pi)^3}\left[(-1)^n - 1\right] = \frac{4}{(n\pi)^3}\left[1-(-1)^n\right].$

Here $b_n = 0 \quad \because g(x) = 0.$

Desired sol. is obtained by substituting a_n & b_n in eq (1).

(b) $u_{tt} = c^2 u_{xx} \quad , \ 0<x<\pi \ , \ t>0$

$u(x,0) = 3\sin x, \quad u_t(x,0) = 0, \quad 0 \le x \le \pi.$

$u(0,t) = u(\pi,t) = 0 \quad , \ t>0.$

Sol. Proceeding as in Q1 (a) part.

Here $a_n = \frac{2}{\pi} \int_0^\pi 3 \sin x \ \frac{\sin n\pi x}{\pi} dx. \quad (n \ne 1)$

$= \frac{6}{2 \times \pi} \int_0^\pi 2 \sin x \sin nx \, dx. \quad [2\sin A \sin B = \cos(A-B) - \cos(A+B)]$

$$= \frac{3}{\pi}\int_0^\pi [\cos(n-1)x - \cos(n+1)x]\,dx \quad (n \neq 1).$$

$$= \frac{3}{\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi = 0.$$

$$a_1 = \frac{2}{\pi}\int_0^\pi 3\sin x \sin x\,dx = \frac{3}{\pi}\int_0^\pi 2\sin^2 x\,dx.$$

$$= \frac{3}{\pi}\int_0^\pi (1-\cos 2x)\,dx = \frac{3}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{3}{\pi}\times\pi = 3.$$

$b_n = 0 \quad \because g(x) = 0.$

$\therefore$ Sol is $u(x,t) = 3\cos\frac{n\pi x}{\pi}t \cdot \sin\frac{n\pi x}{\pi} = 3\cos nxt\sin nx$

2 a) $u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0.$

$u(x,0) = 0, \quad u_t(x,0) = 8\sin^2 x, \quad 0 \le x \le \pi.$

$u(0,t) = 0 = u(\pi,t)$

Sol. Here $f(x) = 0 \quad \therefore \quad a_n = 0.$

$$b_n = \frac{2}{n\pi c}\int_0^\pi 8\sin^2 x \sin\frac{n\pi x}{\pi}\,dx.$$

$$= \frac{8}{n\pi c}\int_0^\pi (2\sin^2 x)\sin nx\,dx$$

$$= \frac{8}{n\pi c}\int_0^\pi (1-\cos 2x)\sin nx\,dx.$$

$$= \frac{8}{n\pi c}\left[-\frac{\cos nx}{n}\Big|_0^\pi - \int_0^\pi \cos 2x \sin nx\,dx\right]$$

$$= \frac{8}{n\pi c}\left[\frac{1}{n}\{1-(-1)^n\}\right] - \frac{4}{n\pi c}\int_0^\pi 2\sin nx\cos 2x\,dx.$$

$$= \frac{8}{n^2\pi c}[1-(-1)^n] - \frac{4}{n\pi c}\int_0^\pi [\sin(n-2)x + \sin(n+2)x]\,dx \quad ; n \neq 2.$$

Note:- for $n = 2$ we shall discuss it separately.

$$a_n = \frac{8}{n^2\pi c}[1-(-1)^n] - \frac{4}{n\pi c}\left[-\frac{\cos(n-2)x}{n-2}\Big|_0^\pi - \frac{\cos(n+2)x}{n+2}\Big|_0^\pi\right]$$

$$= \frac{8}{n^2\pi c}[1-(-1)^n] - \frac{4}{n\pi c}\left[\frac{1}{n-2}\{1-(-1)^{n-2}\} + \frac{1}{n+2}\{1-(-1)^{n+2}\}\right]$$

[Here sign of $(-1)^{n-2}$ & $(-1)^{n+2}$ are same.] equal to $(-1)^n$

$$= \frac{8}{n^2\pi c}[1-(-1)^n] - \frac{4}{n\pi c}\left[\frac{1}{n-2} + \frac{1}{n+2}\right][1-(-1)^n]$$

$$= \frac{8}{n^2\pi c} - \frac{4}{\cancel{n}\pi c}\left(\frac{2\cancel{n}}{n^2-4}\right]\left[1-(-1)^n\right]$$

$$= \frac{8}{\pi c}\left[\frac{1}{n^2} - \frac{1}{n^2-4}\right]\left[1-(-1)^n\right] \quad (A)$$

$$b_n = \frac{8}{\pi c} \times \frac{-4}{n^2(n^2-4)}\left[1-(-1)^n\right] = \frac{32}{\pi c\, n^2(n^2-4)}\left[(-1)^n-1\right] \text{. for } n \neq 2$$

for $n=2$

$$b_2 = \frac{8}{4\pi c}\left[1-(-1)^2\right] - \frac{4}{2\pi c}\int_0^\pi\left[\sin 0x + \sin 4x\right]dx.$$

$$= \overset{\shortparallel}{0} - \frac{2}{\pi c}\int_0^\pi \sin 4x\, dx.$$

$$= \frac{2}{\pi c}\frac{\cos 4x}{4}\Big|_0^\pi$$

$$= \frac{1}{2\pi c}\left[\cos 4\pi - \cos 0\right] = \frac{1}{2\pi c}\left[1-1\right] = 0.$$

Hence the reqd solution would be.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi c}{\pi}t \sin\frac{n\pi x}{\pi}$$ where b_n is given by eq (A).

Non-Homogeneous Problem :- The P.D.E's considered so far are homogeneous. Let us now consider problems involving non-homogeneous equations. We shall again consider initial boundary Value problem.

$$\left.\begin{array}{ll} u_{tt} = c^2 u_{xx} + F(x) \quad , 0<x<l,\ t>0 & \text{—(1.1)} \\ \left.\begin{array}{l} u(x,0) = f(x), \\ u_t(x,0) = g(x), \end{array}\right\} 0 \le x \le l. & \begin{array}{l}\text{—(1.2)} \\ \text{—(1.3)}\end{array} \\ u(0,t) = A \ ,\ u(l,t) = B,\ t>0. & \text{—(1.4)} \end{array}\right] \text{—(1)}$$

We shall assume sol. of eq (1) in the form

$$u(x,t) = v(x,t) + U(x) \quad \text{—(2)}$$

If $u(x,t)$ is a sol. of eq (1) it should satisfy eq (1)

$$\Rightarrow \quad u_{tt} = v_{tt} \quad , \quad u_{xx} = v_{xx} + U_{xx}$$

$$\Rightarrow \quad v_{tt} = c^2(v_{xx} + U_{xx}) + F(x)$$

& if $U(x)$ satisfy the eq. $c^2 U_{xx} + F(x) = 0$ — (3)

then $v(x,t)$ satisfy the wave eq. $v_{tt} = c^2 v_{xx}$

Using eq (2) for $t=0/x=0/x=l$

$$\begin{array}{lll} u(x,0) = v(x,0) + U(x) = f(x) & & \text{from } (1.2) \\ u_t(x,0) = v_t(x,0) = g(x) & & \text{"} \ (1.3) \\ u(0,t) = v(0,t) + U(0) = A, & & \text{"} \ (1.4) \\ u(l,t) = v(l,t) + U(l) = B, & & \text{"} \ (1.4) \end{array}$$

Thus, if $U(x)$ is the solution of the problem

$$\left.\begin{array}{l} c^2 U_{xx} + F(x) = 0, \\ U(0) = A,\ U(l) = B, \end{array}\right] \text{(4)}$$

[Since eq (3) is second order differential equation so we require two initial data to simplify it completely]

then $v(x,t)$ must satisfy

$$\left.\begin{array}{l} v_{tt} = c^2 v_{xx} \\ v(x,0) = f(x) - U(x) \end{array}\right] \text{—(*)}$$

$v_t(x,0) = g(x)$

$v(0,t) = 0 \quad , \quad v(l,t) = 0 . \Bigg] \quad — (*)$

Now simplifying eq ④ we have

$$c^2 U_{xx} + F(x) = 0$$

$$U_{xx} = -\frac{F(x)}{c^2}$$

Integrating we get $\displaystyle U_x = -\frac{1}{c^2}\int_0^x F(\xi)\,d\xi + \underbrace{f(t)}_{\text{const. of integration.}}$

Again Integrating we get.

$$U = \int_0^x \left[-\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi \right] d\eta + f(t)\cdot x + \underbrace{g(t)}_{\text{another const. of integration.}} \quad — ⑤$$

[Note [$\int \sin x\,dx = \int \sin\xi\,d\xi$ etc].

To calculate $f(t)$ & $g(t)$ let us use $U(0) = A$

eq ⑤ $\Rightarrow \quad A = 0 + f(t)\cdot 0 + g(t).$

$\Rightarrow \quad g(t) = A. \quad — ⑥$

eq ⑤ $\Rightarrow U(l) = B \Rightarrow \displaystyle B = \int_0^l \left[-\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi \right] d\eta + f(t)\cdot l + A$

$$\Rightarrow (B-A) + \int_0^l \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi \right] d\eta = f(t)\cdot l \quad — ⑦.$$

Using values from eq ⑥ & ⑦ in eq ⑤ we get.

$$U = \int_0^x \left[-\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi \right] d\eta + (B-A)\frac{x}{l} + \frac{x}{l}\int_0^l \left[\frac{1}{c^2}\int_0^\eta F(\xi)\,d\xi \right] d\eta + A \quad — ⑧$$

This value of U (eq 8) is to be substituted in eq (*). ~~to solve~~ After solving eq (*) for $v(x,t)$, the solution of non-homogeneous wave eq. is given by eq ②

To understand the above article let us consider an example.

Q $\quad u_{tt} = c^2 u_{xx} + h , \quad h$ is a const.

$u(x,0) = 0, \quad u_t(x,0) = 0$

$u(0,t) = 0 \quad , \quad u(l,t) = 0$

Sol:- We assume a sol. in the form
$$u(x,t) = v(x,t) + U(x) \quad -(1)$$
where $U(x)$ satisfies eq. $c^2 U_{xx} + h = 0$, $U(0)=0, U(l)=0$

$\Rightarrow$ Integ. we get $U_x = -\frac{h}{c^2}\cdot x + \underset{c_1 \text{ (say)}}{\text{const}}$

Again integ. we get $U = -\frac{h}{c^2}\cdot\frac{x^2}{2} + c_1 x + \underset{\text{another const. of integ.}}{c_2} \quad -(2)$

Applying boundary conditions we get

(eq 2) $\Rightarrow$ $U(0) = 0 \Rightarrow c_2 = 0$

$U(l) = 0 \Rightarrow 0 = -\frac{hl^2}{2c^2} + c_1 \cdot l \Rightarrow c_1 = \frac{hl}{2c^2}$

eq (2) becomes
$$U(x) = -\frac{hx^2}{2c^2} + \frac{hlx}{2c^2} = \frac{h}{2c^2}(lx - x^2) \quad -(3)$$

Now as illustrate in above article function $v(x,t)$ must satisfy [(eq *) above]
$$v_{tt} = c^2 v_{xx}.$$
$$v(x,0) = 0 - \frac{h}{2c^2}(lx - x^2), \quad v_t(x,0) = 0$$
$$v(0,t) = 0, \quad v(l,t) = 0$$

Now sol. of wave eq is given by
$$v(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l} \quad -(4)$$

Here $b_n = 0$ ($\because g(x) = 0$)

& $a_n = \frac{2}{l}\int_0^l \left[-\frac{h}{2c^2}(lx - x^2)\right]\sin\frac{n\pi x}{l}\,dx$

$$a_n = \frac{2}{l}\left[-\frac{h}{2c^2}\left\{(lx-x^2)\cos\frac{n\pi x}{l}\times\frac{l}{n\pi}\Big|_0^l - \int_0^l (l-2x)\times\cos\frac{n\pi x}{l}\times\frac{l}{n\pi}\,dx\right\}\right.$$

Again integrating by parts

$$= 0 + \frac{h}{n\pi c^2}\left[\frac{l}{n\pi}(l-2x)\sin\frac{n\pi x}{l}\Big|_0^l - \int_0^l -2\sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\,dx\right]$$

$$= 0 + \frac{hl}{(n\pi c)^2}\left[-2\cos\frac{n\pi x}{l}\cdot\frac{l}{n\pi}\right]_0^l = \frac{-2hl^2}{(n\pi)^3 c^2}\left[1-(-1)^n\right]$$

$\because \cos n\pi = (-1)^n$

$\therefore \quad a_n = \begin{cases} -\dfrac{4l^2h}{(n\pi)^3c^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$

The sol. of the given I.B.V.P. is $\therefore$, given by.

$u(x,t) = v(x,t) + U(x)$

where $v(x,t)$ is given by eq (4) & $U(x)$ is given by eq (3).

Now let us discuss few more examples from ex (7.9) to understand above concepts properly. Let Now we shall take.

Q3(b) $\quad u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < \pi, t > 0$ (10)

$u(x,0) = \sin x \quad , 0 \le x \le \pi$

$u_t(x,0) = x^2 - \pi x \quad , 0 \le x \le \pi$

$u(0,t) = u(\pi,t) = 0, \; t > 0$

The solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{l}\right)t + b_n \sin\left(\frac{n\pi c}{l}\right)t\right] \sin\frac{n\pi x}{l} \quad \text{—(1)}$$

where $a_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$, $b_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$

Here $f(x) = \sin x$, $g(x) = x^2 - \pi x$, $l = \pi$

Now
$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin\frac{n\pi x}{\pi}\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\left[\cos(1-n)x - \cos(1+n)x\right]dx$$
$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$$
$$= 0 \quad \forall \; n \neq 1$$

and
$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$$
$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{\pi}\cdot\pi = 1$$

$\therefore a_n = \begin{cases} 0 & \forall\, n \neq 1 \\ 1 & , n = 1 \end{cases}$

Now
$$b_n = \frac{2}{n\pi c}\int_0^{\pi}(x^2 - \pi x)\sin nx\,dx, \quad l = \pi$$
$$= \frac{2}{n\pi c}\left[-\left(\frac{(x^2-\pi x)\cos nx}{n}\right)\Big|_0^{\pi} + \int_0^{\pi}\frac{(2x-\pi)\cos nx}{n}\,dx\right.$$
$$= \frac{2}{n\pi c}\left[\frac{1}{n}\left\{\frac{(2x-\pi)\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{2\sin nx}{n}\,dx\right.\right.$$

$$= \frac{2}{n\pi c}\left[\frac{1}{n}\cdot\frac{x\cos nx}{n^2}\right]_0^{\pi} = \frac{4}{n^2\pi c}\left(\cos n\pi - 1\right)$$ (11)

ie. $$b_n = \frac{4[(-1)^n - 1]}{n^2\pi c} \quad \forall \; n$$

and $$b_1 = -\frac{8}{\pi c}$$

Hence the solution is given by

$$u(x,t) = \left(\cos ct - \frac{8}{\pi c}\sin ct\right)\sin x + \sum_{n=2}^{\infty}\frac{4[(-1)^n - 1]}{n^2\pi c}\sin(nct)\sin nx.$$

Q11) $u_{tt} = c^2 u_{xx} + A\sinh x \;, \quad 0 < x < l, \; t > 0$

$u(x,0) = 0 \;, \quad u_t(x,0) = 0 \quad 0 \le x \le l$

$u(0,t) = h, \; u(l,t) = k \;, \quad t > 0 \;, \quad h, k, A$ are constants

Let us assume that

$$u(x,t) = v(x,t) + U(x) \qquad \text{—— (1)}$$

Then eqn reduces to $v_{tt} = c^2(v_{xx} + U_{xx}) + A\sinh x$.

Using the initial ~~condition~~ and boundary conditions

$u(x,0) = v(x,0) + U(x) = 0, \quad u_t(x,0) = v_t(x,0) = 0$

$\Rightarrow \quad v(x,0) = -U(x) \;, \; \& \; v_t(x,0) = 0$

$u(0,t) = v(0,t) + U(0) = h \quad \& \quad u(l,t) = v(l,t) + U(l) = k$

$\Rightarrow \quad v(0,t) = h - U(0)$

$\quad\quad v(l,t) = k - U(l)$

Thus if $U(x)$ is the soln of the problem

$$c^2 U_{xx} + A\sinh x = 0$$

$$U(0) = h, \quad U(l) = k$$

then $v(x,t)$ must satisfy

$v_{tt} = c^2 v_x$

$v(x,0) = -U(x)$

$v_t(x,0) = 0$

$v(0,t) = v(l,t) = 0$

Now $c^2 U_{xx} + A\sinh x = 0 \Rightarrow U_x = -\frac{A}{c^2}\cosh x + c_1$

$$\therefore U = -\frac{A}{c^2}\sinh x + c_1 x + c_2$$

using $U(0) = h$, $U(l) = k$, we get $h = c_2$

and $c_1 = \frac{k-h}{l} + \frac{A}{lc^2}\sinh l.$

$\therefore\ U = -\frac{A}{c^2}\sinh x + \left(k - h + \frac{A}{c^2}\sinh l\right)\frac{x}{l} + h$ (12)

Now sol^n of v is given by

$$v(x,t) = \sum_{n=1}^{\infty}\left\{a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$$

Since $g(x) = 0 \Rightarrow b_n = 0$

and $f(x) = -U(x) = \frac{A}{c^2}\sinh x - \left(\frac{A}{c^2}\sinh l + k - h\right)\frac{x}{l} - h$

$$\therefore\ a_n = \frac{-2}{l}\int_0^l U(x)\sin\frac{n\pi x}{l}\,dx$$

$$\therefore\ v(x,t) = \sum_1^{\infty}\left[-\frac{2}{l}\int_0^l U(\tau)\sin\frac{n\pi \tau}{l}\,d\tau\right]\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$\therefore\ u(x,t) = v(x,t) + U(x)$ where

$v(x,t) = ?$

$U(x) = ?$

~~Q.5(c)~~ Q13) $u_{tt} = c^2 u_{xx} + x^2, \quad 0 < x < 1,\ t > 0$ (2)

$u(x,0) = x \quad , 0 \le x \le 1$

$u_t(x,0) = 0 \quad , 0 \le x \le 1$

$u(0,t) = 0,\ u(1,t) = 1,\ t > 0$

Let $u(x,t) = v(x,t) + U(x)$

Eq^n reduces to $v_{tt} = c^2(v_{xx} + U_{xx}) + x^2$

with

$$u(x,0) = v(x,0) + U(x) = x$$
$$u_t(x,0) = v_t(x,0) = 0$$
$$u(0,t) = v(0,t) + U(0) = 0$$
$$u(1,t) = v(1,t) + U(1) = 1$$

If $U(x)$ satisfies the eq^n

$$c^2 U_{xx} + x^2 = 0, \quad U(0) = 0, \quad U(1) = 1$$

i.e. $U = -\dfrac{x^4}{12c^2} + k_1 x + k_2$

$k_2 = 0, \quad k_1 = \dfrac{1}{12c^2} + 1 \Rightarrow U = -\dfrac{x^4}{12c^2} + \left(1 + \dfrac{1}{12c^2}\right)x$

Then $v(x,t)$ must satisfy the wave eq^n

$$v_{tt} = c^2 v_{xx}$$
$$v(x,0) = x - U(x)$$
$$v_t(x,0) = 0$$
$$v(0,t) = v(1,t) = 0$$

$\therefore$ the sol^n $v(x,t)$ is given by

$$\sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi c}{l}t + b_n \sin\frac{n\pi c}{l}t\right)\sin\frac{n\pi x}{l}$$

where $a_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \quad b_n = \dfrac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$

here $g(x) = 0, \therefore b_n = 0, \quad l = 1, \quad f(x) = \cancel{x} + \dfrac{x^4}{12c^2} - (\cancel{1} + \dfrac{1}{12c^2})x$

$$= \frac{x^4}{12c^2} - \frac{x}{12c^2}$$

$$\therefore a_n = 2\int_0^1\left(\frac{x^4}{12c^2} - \frac{x}{12c^2}\right)\sin n\pi x\,dx$$

$$= \frac{1}{6c^2}\left[\left|-\frac{(x^4-x)\cos n\pi x}{n\pi}\right|_0^1 + \int_0^1 \frac{(4x^3-1)\cos n\pi x}{n\pi}dx\right.$$

$$= \frac{1}{6c^2 n\pi}\left[\left|\frac{(4x^3-1)\sin n\pi x}{n\pi}\right|_0^1 - \int_0^1 \frac{12x^2\sin n\pi x}{n\pi}dx\right]$$

$$= \frac{12}{6c(n\pi)^2}\left[\left|\frac{x^2\cos n\pi x}{n\pi}\right|_0^1 + \int_0^1 -2x\frac{\cos n\pi x}{n\pi}dx\right]$$

$$= \frac{2}{c^2(n\pi)^3}\left[\cos n\pi - 2\left|\frac{x\sin n\pi x}{n\pi}\right|_0^1 + \int_0^1\frac{\sin n\pi x}{n\pi}dx\right]$$

$$= \frac{2}{c^2(n\pi)^3}\left[(-1)^n + 2\left|\frac{\cos n\pi x}{(n\pi)^2}\right|_0^1\right]$$

$$= \frac{2}{c^2(n\pi)^3}\left[(-1)^n + \frac{2}{(n\pi)^2}\left(\cos n\pi - 1\right)\right]$$

$$= \frac{2}{c^2(n\pi)^3}\left[(-1)^n + \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)\right]$$

$$\therefore\ v(x,t) = \sum_1^\infty \frac{2}{c^2(n\pi)^3}\left\{(-1)^n + \frac{2}{(n\pi)^2}\left((-1)^n-1\right)\right\} \cos n\pi c t \, \sin n\pi x$$

$\therefore \quad u(x,t) = v(x,t) + U(x)$ where $v(x,t) = \ldots$

$U(x) = \ldots$

~~Q8(a)~~ Q7+

Q15 d) $u_t = k u_{xx}, \quad 0<x<l, \; t>0$ (15)

$u(0,t) = 0 \quad, t \geq 0$

$u(l,t) = 1 \quad, t \geq 0$

$u(x,0) = \sin\frac{\pi x}{2l} \quad, 0 \leq x \leq l$

let $u(x,t) = v(x,t) + \frac{x}{l}$.

eqn reduces to $v_t = k v_{xx}$

$v(x,0) = u(x,0) - \frac{x}{l} = \sin\frac{\pi x}{2l} - \frac{x}{l} = f(x)$

$v(0,t) = 0, \quad v(l,t) = 0$

which is the heat equation and the solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l}$$

where $a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$.

$$= \frac{2}{l}\int_0^l \left[\sin\left(\frac{\pi x}{2l}\right) - \frac{x}{l}\right] \sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{1}{l}\int_0^l 2\sin\frac{\pi x}{2l}\sin\frac{n\pi x}{l}\,dx - \frac{2}{l^2}\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\cos\left(\frac{\pi x}{2l} - \frac{n\pi x}{l}\right) - \cos\left(\frac{\pi x}{l} + \frac{n\pi x}{2l}\right)\right]dx - \frac{2}{l^2}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l} I_1 - \frac{2}{l^2} I_2$$

$$I_1 = \int_0^l \left[\cos\left(\tfrac{1}{2}-n\right)\frac{\pi x}{l} - \cos\left(\tfrac{1}{2}+n\right)\frac{\pi x}{l}\right]dx$$

$$= \left|\frac{\sin\left(\frac{1}{2}-n\right)\frac{\pi x}{l}}{\left(\frac{1}{2}-n\right)\frac{\pi}{l}}\right|_0^l - \left|\frac{\sin\left(\frac{1}{2}+n\right)\frac{\pi x}{l}}{\left(\frac{1}{2}+n\right)\frac{\pi}{l}}\right|_0^l$$

$$= \frac{l\sin\left(\frac{1}{2}-n\right)\pi}{\left(\frac{1}{2}-n\right)\pi} - \frac{l\sin\left(\frac{1}{2}+n\right)\pi}{\left(\frac{1}{2}+n\right)\pi}$$

$$= \frac{l}{\pi}\left[\frac{\sin\left(\frac{\pi}{2}-n\pi\right)}{\left(\frac{1}{2}-n\right)} - \frac{\sin\left(\frac{\pi}{2}+n\pi\right)}{\left(\frac{1}{2}+n\right)}\right]$$

$$= \frac{l}{\pi}\left[\frac{2\cos n\pi}{1-2n} - \frac{2\cos n\pi}{1+2n}\right]$$

$$= \frac{2l}{\pi}(-1)^n\left[\frac{2n+1-1+2n}{1-4n^2}\right] = \frac{8ln(-1)^n}{\pi(1-4n^2)}$$

$$I_2 = \int_0^l x \sin\frac{n\pi x}{l}\,dx$$

$$= \left|\frac{xl}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right|_0^l + \int_0^l \frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx$$

$$= -\frac{l^2}{n\pi}\cos n\pi + \left(\frac{l}{n\pi}\right)^2 \left|\sin\frac{n\pi x}{l}\right|_0^l$$

$$= -\frac{l^2}{n\pi}(-1)^n$$

$$\therefore\ a_n = \frac{1}{l}\,\frac{8nl(-1)^n}{\pi(1-4n^2)} - \frac{2}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right]$$

$$= (-1)^n\left[\frac{8n}{\pi(1-4n^2)} + \frac{2}{n\pi}\right]$$

$$= \frac{2(-1)^n}{n\pi(1-4n^2)}$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi(1-4n^2)}\, e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi(1-4n^2)}\, e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi x}{l} + \frac{x}{l}$$

Q5(b) $\quad u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l,\ t > 0$

$$u(x,0) = f(x), \quad 0 \le x \le l$$

$$u_t(x,0) = g(x), \quad 0 \le x \le l$$

$$u(0,t) = u(l,t) = 0, \quad t > 0$$