This question paper contains 3 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : 32

Unique Paper Code : 235566 G

Name of the Paper : **MAPT-505, Real Analysis**

Name of the Course : **B.Sc. Mathematical Sciences/B.Sc. Physical Sciences**

Semester : **V**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Attempt any *two* parts from each question.

1. (*a*) State order completeness property. Show that the set of irrational numbers is not order complete.

   (*b*) State and prove Archimedean property of real numbers.

   (*c*) Show that the set of positive rational numbers is countable. 6,6

2. (*a*) Prove that the set $S = \{1/n | n \in N\}$ has no limit point other than zero in the set of real numbers.

   (*b*) Show that every subset of a countable set is countable. Is every superset of a countable set countable ? Justify.

   (*c*) If $\lim a_n = a$ and $a_n \geq 0$ for all n, then prove that $a \geq 0$. Deduce that if $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n \geq b_n$ for all n, then $\lim a_n \geq \lim b_n$. 6,6

P.T.O.

3. (*a*) State and prove Bolzano-Weierstrass theorem for sequences.

(*b*) Define a Cauchy sequence. Show that the sequence $\{a_n\}$ is not convergent, where :

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots .$$

(*c*) Let $\{x_n\}$ be a sequence defined by :

$$x_1 = 1,\ x_{n+1} = \frac{3 + 2a_n}{2 + a_n},\ n \geq 1.$$

Show that the sequence $\{x_n\}$ is convergent. Also find its limit. 6½,6½

4. (*a*) Define the convergence of an infinite series. Show that the series :

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots$$

is convergent.

(*b*) Test the convergence of the following series :

(*i*) $\sum \frac{\sqrt{n+1} - \sqrt{n-1}}{n}$

(*ii*) $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots$ for all $x > 0$.

(*c*) Let $\Sigma\, u_n$ be a positive term series such that

$$\lim_{n \to \infty} (u_n)^{\frac{1}{n}} = \text{L}.$$

Show that $\Sigma\, u_n$ converges for $\text{L} < 1$. What happens for $\text{L} = 1$? Justify. 6,6

5. (*a*) State Leibnitz test for convergence of an alternating series. Test for convergence and absolute convergence, the series :

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots .$$

Is the series conditionally convergent ?

(*b*) Define radius of convergence and interval of convergence of a power series. Find the radius of convergence and exact interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}.$$

(*c*) Define sine function $S(x)$ and cosine function $C(x)$ in terms of power series. Specify the domains of convergence of the respective power series. 6½, 6½

6. (*a*) If $\{f_n\}$ is a sequence of continuous functions converging uniformly to a function f on $[a, b]$, then show that f is continuous on $[a, b]$.

(*b*) Test the sequence $\{f_n\}$ where

$$f_n(x) = \frac{nx}{1+n^2x^2}$$

for uniform convergence on $[0, 1]$. Verify :

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1\left(\lim_{n\to\infty} f_n(x)\right)dx.$$

(*c*) (*i*) Show that the series :

$$\sum \frac{1}{n^4+n^2x^2}$$

converges uniformly for all real values of x.

(*ii*) Show that the series :

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} + \frac{x^4}{\left(1+x^4\right)^3} + \ldots$$

is not uniformly convergent on $[0, 1]$. 6½,6½

Q No 1(a) order completeness property (1)

Let S be a non-empty subset of $\mathbb{R}$, the set of real numbers, S is said to be complete if every non-empty subset of S that has an upper bound also has a supremum in S.

We show that the set of all irrational number is not order complete.

Pf: Q^c is set of all irrational numbers.

Define $S = \{a_n \mid n = 1, 2, \ldots\}$ & $a_n = \sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$ n terms

S is a non-empty subset of R

We show that S is bounded above,

We show that $1 < a_n < 2$ $\forall n$ by induction on n.

For $n=1$, $1 < a_1 = \sqrt{2} < 2$ $\therefore$ result is true in this case.

Assume the result is true for $n = k$. Then $1 < a_k < 2$

We prove the result for $n=k+1$.

$$a_{k+1} = \sqrt{2+a_k} < \sqrt{2+2} = 2$$

Also $a_{k+1} = \sqrt{2+a_k} > \sqrt{2+1} = \sqrt{3} > 1$

Thus $1 < a_{k+1} < 2$

By induction $1 < a_n < 2$ $\forall n$.

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Thus S is ~~bdd~~ bounded above.

We prove that $\sup S = 2 \notin Q^{c}$. (2)

For this we firstly prove that $a_{n+1} > a_n \ \forall n$

$$a_{n+1} = \sqrt{2+\sqrt{2+\sqrt{2+\dots}}} = \sqrt{2+a_n}$$

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n = \frac{(\sqrt{2+a_n} - a_n)(\sqrt{2+a_n} + a_n)}{(\sqrt{2+a_n} + a_n)}$$

$$= \frac{2+a_n - a_n^2}{\sqrt{2+a_n} + a_n} = -\frac{(a_n-2)(a_n+1)}{\sqrt{2+a_n} + a_n} > 0$$

(Since $1 < a_n < 2$)

Thus $a_{n+1} > a_n$.

Thus seq $\langle a_n \rangle$ is monotonically increasing

As a seq of real numbers, $\langle a_n \rangle$ is m. i & bdd above,

Hence it cgs to its supremum

Let $\lim_{n\to\infty} a_n = l$.

Now $a_{n+1} = \sqrt{2+a_n}$ implies $a_{n+1}^2 = 2+a_n$

Thus $\lim_{n\to\infty} a_{n+1}^2 = 2 + \lim_{n\to\infty} a_n$. That is $\left(\lim_{n\to\infty} a_{n+1}\right)^2 = 2 + \lim_{n\to\infty} a_n$

i.e. $l^2 = 2 + l$

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$l^2 - l - 2 = 0$

$l = \frac{1 \pm \sqrt{1-4(-2)}}{2} = \frac{1 \pm 3}{2} = 2$ or -1 (2)

Since $a_n \geq 1$, $l = \lim_{n\to\infty} a_n \geq 1$

Thus $l = 2$

Hence $\sup a_n = \lim_{n\to\infty} a_n = 2 \notin Q^*$

Thus the set S does not have its supremum in Q^*

Hence Q^* is not complete.

Note: Above Q may be asked like

Discuss cgs of the seq $\langle a_n \rangle$ defined by

$a_1 = 1$

$a_{n+1} = \sqrt{2 + a_n}$, $n \geq 1$

Show that $\lim_{n\to\infty} a_n = 2$

(b) Statement of Archimedean property of real numbers.

If $x \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ s.t $x < n$

where $\mathbb{R}$ is set of real numbers & $\mathbb{N}$ is set of natural numbers.

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Proof of Archimedean property

Suppose there is no $n \in \mathbb{N}$ such that $x < n$.

Then $n \leq x \quad \forall \ n \in \mathbb{N}$.

Thus x is an upper bound of $\mathbb{N}$, By Completeness property of $\mathbb{R}$, $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. That is

$$\sup \mathbb{N} = u$$

Since $u-1 < u$, $u-1$ is not an upper bound of $\mathbb{N}$.

Hence $u-1 < m$ for some $m \in \mathbb{N}$.

Thus $u < m+1$ & $m+1 \in \mathbb{N}$

which contradict that u is an upper bound of $\mathbb{N}$.

Hence there exists $n \in \mathbb{N}$ s.t $x < n$.

(c) Show that the set of positive rational numbers is countable

Pf Consider the set

$$A = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N} \right\}$$

Note: $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}$ are consider different element in A.

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Define $f: N \times N \to A$ by
$f(m,n) = \frac{m}{n}$

(5)

<u>f is well-define</u>

Let $(m,n), (m',n') \in N \times N$ s.t $(m,n) = (m',n')$. Then $m = m'$ & $n = n'$. Hence $\frac{m}{n} = \frac{m'}{n'}$.

$\therefore f(m,n) = f(m',n')$

f is well-define

<u>f is 1-1</u> Let $(m,n), (m',n') \in N \times N$ s.t $f(m,n) = f(m',n')$

Then $\frac{m}{n} = \frac{m'}{n'}$ in A

$\Rightarrow m = m'$ & $n = n'$ in A (Note: $\frac{m}{n}$, $\frac{2m}{2n}$ are different elements in A)
ie $\frac{m}{n} \neq \frac{2m}{2n}$ in A.

Thus $(m,n) = (m',n')$

$\therefore$ Hence f is 1-1

<u>f is onto</u> Let $\frac{m}{n} \in A$. By definition of f,

$\frac{m}{n} = f(m,n)$

$\therefore$ f is onto.

Hence f is 1-1 ~~and~~ mapping from $N \times N$ onto A.

Since $N \times N$ is countable, A is countable. Q^+, the set of all positive rational numbers is a subset of countable set A. Since every subset of a countable set is countable, Q^+ is countable. (6)

Q 2(a) Show that the set $S = \{\frac{1}{n} \mid n \in N\}$ has no limit point other that zero in the set of real numbers.

Pf Firstly we recall def of a limit pt.

l is called a limit pt of a subset A of real numbers if for every $\epsilon > 0$, $\left( ]l-\epsilon, l+\epsilon[- \{l\} \right) \cap A \neq \phi$

$]\ l\ [$
$l-\epsilon \quad l \quad l+\epsilon$

Proof of main part

We show that '0' is a limit pt of S.

Let $\epsilon > 0$ be given. By Archimedean property, there is a $n \in N$ s.t

$$\frac{1}{\epsilon} < n.$$

$\frac{1}{n}$
$-\epsilon \quad 0 \quad \epsilon$

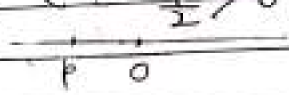
i.e. $\epsilon > \frac{1}{n} > 0$

$\therefore \frac{1}{n} \in \left( ]0-\epsilon, 0+\epsilon[- \{0\} \right) \cap S$ for every $\epsilon > 0$.

$\therefore$ 0 is a limit pt of S.

Let p be any real number other than 0. We show that p is not a limit pt of S.

Case (i) $p<0$. ~~For every $\epsilon>0$ with~~ Take $\epsilon = -\frac{p}{2} > 0$.



Then $]p-\epsilon, p+\epsilon[\; = \;]p+\frac{p}{2}, p-\frac{p}{2}[$

$= \;]\frac{3p}{2}, \frac{p}{2}[$

& $(]p-\epsilon, p+\epsilon[- \{p\}) \cap S = (]\frac{3p}{2}, \frac{p}{2}[- \{p\}) \cap S = \Phi$.

$\therefore$ p is not a limit pt of S

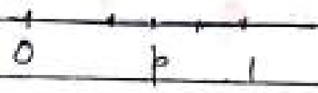
Case (ii) $p>1$ Take $\epsilon = \frac{p-1}{2}$. Then $p-\epsilon = p - \frac{p-1}{2} = \frac{p+1}{2} < 1$

Thus $]p-\epsilon, p+\epsilon[\cap S = \Phi$.

Hence p is not a limit pt

Case (iii) $0<p<1$ & $p \notin S$. Then there exists a positive integer m s.t. $m < \frac{1}{p} < m+1$

i.e. $\frac{1}{m+1} < p < \frac{1}{m}$



Thus $]\frac{1}{m+1}, \frac{1}{m}[\cap S = \phi$

Hence p is not a limit pt of S

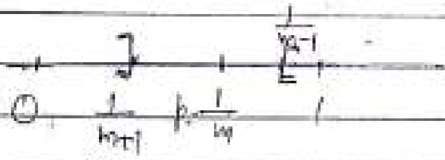
Case (iv) $p = 1$, Then take $\epsilon = \frac{1}{2}$,

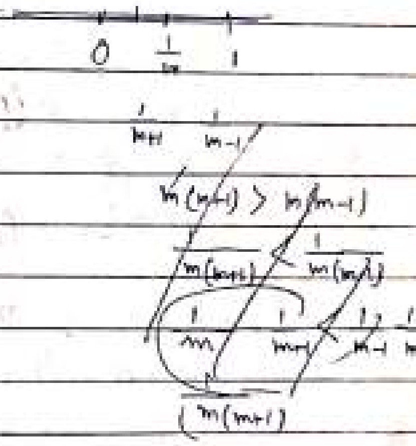
$$\left( ]p-\epsilon, p+\epsilon[- \{p\} \right) \cap S = \left( ]\tfrac{1}{2}, \tfrac{3}{2}[- \{1\} \right) \cap S = \phi$$

Thus $p = 1$ is not a limit pt of S

Case (v) $p \neq 1$ & $p \in S$. Then let $p = \frac{1}{m}$, for some positive integer m.

Take ϵ $\quad$ Then $\frac{1}{m+1} < \frac{1}{m} < \frac{1}{m-1}$





Then $\left( ]\frac{1}{m+1}, \frac{1}{m-1}[- \{\frac{1}{m}\} \right) \cap S = \phi$

Thus $\frac{1}{m}$ is not a limit pt of S

i.e. no pt of S is a limit pt of S.

Hence '0' is the only limit pt of S.

2 (b) S Show that every subset of a countable set is countable. (9)
Is every superset of a countable set countable? Justify.

Pf. Let A be a countable set and let B be a subset of A. If B is finite, then there is nothing to prove. Therefore we may assume B is infinite subset of A. The set A is also infinite countable set.

let $A = \{a_1, a_2, \ldots\}$

Since B is subset of A, $a_i \in B$ for some i.

Choose $a_i \in B$ with smallest subscript.

Let n_1 be the smallest positive integer s.t $a_{n_1} \in B$.

Consider $A - \{a_{n_1}\}$. Then $B - \{a_{n_1}\} \subseteq A - \{a_{n_1}\}$

Choose n_2 the smallest positive integer s.t $a_{n_2} \in B - \{a_{n_1}\} \subseteq A$

Continue this process, we get a sequence $\{n_k\}$ with

$n_1 < n_2 < \ldots$ and $a_{n_i} \in B \quad \forall i = 1, 2, \ldots$

Define $f : N \to B$ by

$$f(i) = a_{n_i}, \quad i = 1, 2, 3,$$

Then f is 1-1 & onto.

Hence B is countable.

~~Every superset of a countable set is not countable,~~

~~$\mathbb{Q}$, the set of rational numbers is countable.~~

~~$\mathbb{R}$ is superset of $\mathbb{Q}$, $\mathbb{R}$ is uncountable,~~

Q2(c) If $\lim\limits_{n\to\infty} a_n = a$ & $a_n \geq 0 \ \forall n$, then prove that $a \geq 0$.

Deduce that if $\{a_n\}$ and $\{b_n\}$ are two sequences s.t. $a_n \geq b_n \ \forall n$, then $\lim\limits_{n\to\infty} a_n \geq \lim\limits_{n\to\infty} b_n$

Pf. Let $\lim\limits_{n\to\infty} a_n = a$ & $a_n \geq 0 \ \forall n$.

Claim $a \geq 0$.

If possible, this is not the case. Then $a < 0$.

Take $\epsilon = -\frac{a}{2} > 0$. Since $\lim\limits_{n\to\infty} a_n = a$, for $\epsilon = -\frac{a}{2} > 0$, there exists a positive integer N s.t

$$|a_n - a| < \epsilon \quad \forall n \geq N$$

i.e $a - \epsilon < a_n < a + \epsilon \quad \forall n \geq N$

$$\Rightarrow a_n < a + \epsilon = a - \frac{a}{2} = \frac{a}{2} < 0 \quad \forall n \geq N$$

Then $a_n < 0 \quad \forall n \geq N$

Which contradict that $a_n \geq 0 \quad \forall n$ (11)

Hence $a \geq 0$.

2nd part Let $a_n \geq b_n \quad \forall n$, Then $c_n = a_n - b_n \geq 0 \quad \forall n$

Then $\lim_{n\to\infty} c_n \geq 0$ by above result

Hence $\lim_{n\to\infty} (a_n - b_n) \geq 0$

i.e. $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$.

Q.No 3(a) State and prove Bolzano Weierstrass Th for sequence

Pf Statement of Bolzano Weierstrass Th for sequence

'Every bounded sequence has a limit point.

proof of Th Let $\langle a_n \rangle$ be a bounded sequence.

Let $S = \{a_n \mid n \in N\}$

Since ~~as~~ sequence $\langle a_n \rangle$

Case(i) S is infinite set

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By Bolzano Weirstrass Th, S has a limit point, p (say). Since ~~S has~~ p is a limit pt of S, every neighbourhood of p contains infinitely many pts of S. This means every ~~nbd~~ neighbourhood of p contains infinitely many points of $\langle a_n \rangle$

i.e every neighbourhood of p $N_\delta(p)$ of p, there is a positive integer m s.t

$$a_{n_k} \in N_\delta(p) \quad \forall \; n_k > m$$

where $N_\delta(p) =]p-\delta, p+\delta[$, $\delta > 0$.

Thus p is a limit pt of the sequence $\langle a_n \rangle$.

Case (ii) S is a finite set.. Then

$$a_n = a \text{ for infinitely many } n.$$

Clearly 'a' is a limit pt of S.

3 (b) Sol: Define a Cauchy's seq. Show that the sequence $\{a_n\}$ is not a Cauchy sequence, where

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1}$$

Pf. ~~ans~~ A sequence $\langle a_n \rangle$ is said to be Cauchy seq if (13)
for every $\epsilon > 0$, $\exists$ a positive integer N s.t

$$|a_n - a_m| < \epsilon \quad \forall\ n, m \geq N,$$

2nd part — We show that $\langle a_n \rangle$ define by

$$a_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1}$$

is not a Cauchy sequence.

If possible, $\langle a_n \rangle$ is cauchy sequence.

Take $\epsilon = \frac{1}{4}$. Then for $\epsilon = \frac{1}{4} > 0$ $\exists$ a positive integer N s.t

$$|a_n - a_m| < \epsilon = \frac{1}{4} \quad \forall\ n, m \geq N$$

Take $n = 2N$, $m = N$. Then

$$|a_{2N} - a_N| < 1/4 \quad \text{———} \quad (1)$$

Now $a_N = 1 + \frac{1}{3} + \frac{1}{5} + \text{——} + \frac{1}{2N-1}$

$$a_{2N} = \underbrace{1 + \frac{1}{3} + \frac{1}{5} + \text{——} + \frac{1}{2N-1}} + \frac{1}{2N+1} + \frac{1}{2N+3} + \cdots + \frac{1}{4N-1}$$

Hence (1) implies

$$\left| 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2N-1} + \left(\frac{1}{2N+1} + \frac{1}{2N+3} + \cdots + \frac{1}{4N-1}\right) - \left(1 + \frac{1}{3} + \cdots + \frac{1}{2N-1}\right)\right| < 1/4$$

ie $\frac{1}{2N+1} + \frac{1}{2N+3} + \text{——} + \frac{1}{4N-1} \leq 1/4$

$$\frac{1}{4} > \underbrace{\frac{1}{2N+1} + \frac{1}{2N+3} + \cdots + \frac{1}{4N-1}}_{N \text{ term}} > \underbrace{\frac{1}{4N} + \frac{1}{4N} + \cdots + \frac{1}{4N}}_{N \text{ term}} = \frac{N}{4N} \geq \frac{1}{4}$$

which is a contradict

Thus $\langle a_n \rangle$ is not a cauchy sequence

3 (c) Let $\{x_n\}$ be a sequence defined by

$$x_1 = 1, \quad x_{n+1} = \frac{3+2x_n}{2+x_n}, \quad n \geq 1$$

Show that the seq $\langle x_n \rangle$ is a cgt seq. Also find its limit

Firstly we show that

$$x_n > 0 \quad \forall\, n$$

by induction on n,

For $n=1$, $x_1 = 1 > 0$ $\therefore$ result is true in this case.

Assume the result is true for $n=k$, Then $x_k > 0$.

Now we prove the result for $n=k+1$.

We have

$$x_{k+1} = \frac{3+2x_k}{2+x_k} > 0$$

Thus $x_n > 0 \ \forall n$ by induction

Now we show that $\{x_n\}$ is bdd above, For this we show that

$x_n < 2 \ \forall n$,

Clearly $x_1 = 1 < 2$,

For any $n \geq 1$,

$$x_{n+1} - 2 = \frac{3+2x_n}{2+x_n} = \frac{3+2x_n}{2+x_n} - 2 = \frac{3+2x_n - 4 - 2x_n}{2+x_n}$$

$$= -\frac{1}{(2+x_n)} < 0 \quad (\text{Since } x_n > 0 \ \forall n)$$

Thus $x_{n+1} < 2 \quad \forall n \geq 1$

Hence $\{x_n\}$ is bdd above

Now we show that $\{x_n\}$ is monotonically increasing

We write $u_n = x_{n+1} - x_n \quad \forall n = 1, 2, \ldots$

We prove $u_n > 0 \ \forall n$ by induction on n

For $n = 1$, $u_1 = x_2 - x_1 = \frac{3+2}{2+1} - 1 = \frac{2}{3} > 0$

Suppose $u_k > 0$. We show that $u_{k+1} > 0$

$u_k = x_{k+1} - x_n$ & $u_{k+1} = x_{k+2} - x_{n+1} = \frac{3+2x_{k+1}}{2+x_{k+1}} - \frac{3+2x_k}{2+x_k}$

$$= \frac{6 + 3x_k + 2x_{k+1} + x_{k+1}x_k - (6 + 2x_k + 3x_{k+1} + x_{k+1}x_k)}{(2+x_{k+1})(2+x_k)}$$

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$$= \frac{(3+x_{k+1})(2+x_k) - (3+2x_k)(2+x_{k+1})}{(2+x_{k+1})(2+x_k)} =$$

$$= \frac{6+3x_k+4x_{k+1}+2x_{k+1}x_k - (6+3x_{k+1}+4x_k+2x_{k+1}x_k)}{(2+x_{k+1})(2+x_k)}$$

$$= \frac{(x_{k+1}-x_k)}{(2+x_{k+1})(2+x_k)} = \frac{y_k}{(2+x_{k+1})(2+x_k)} > 0$$

Thus by induction,

$y_n > 0 \ \forall n$

i.e. $x_{n+1} - x_n > 0 \ \forall n$

i.e. $\{x_n\}$ is monotonically increasing

By Monotone cgt Th $\{x_n\}$ is cgt

Let $\lim_{n\to\infty} x_n = l$. Then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{3+2x_n}{2+x_n} = \frac{3+2\lim x_n}{2+\lim x_n}$$

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Then $l = \frac{3+2l}{2+l}$

$\Rightarrow l^2+2l = 3+2l$

$\Rightarrow l^2 = 3$

$\Rightarrow l = \pm\sqrt{3}$

Since $0 < x_n < 2$

$0 \leq \lim x_n \leq 2$

i.e $0 \leq l \leq 2$

Hence $l = \sqrt{3}$