29/04/2020 Lecture - 32

Q. find the least - square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Also verify that the least solution $v$ satisfies $\|Av - b\| \le \|Az - b\|$ for the vector $z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Soln To find a least - square solution for the system $Ax = b$, we need to solve the linear system

$$(A^T A)x = A^T b$$

Now $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$$(A^T A) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$$

$$(A^TA) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+4 & 4 \\ 4 & 5 \end{bmatrix}$$

$$(A^TA) = \begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix}$$

Now $A^Tb = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+6 \\ 2+6 \end{bmatrix}$

$$A^Tb = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Now $(A^TA)x = A^Tb$

$$\begin{bmatrix} 8 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Augmented matrix for this system is

$$\left[\begin{array}{cc|c} 8 & 4 & 8 \\ 4 & 5 & 8 \end{array}\right] \xrightarrow{R_1 \to R_1/8} \left[\begin{array}{cc|c} 1 & 1/2 & 1 \\ 4 & 5 & 8 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$ $\qquad$ $R_2 \to R_2/3$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 1 \\ 0 & 3 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1/2 & 1 \\ 0 & 1 & 4/3 \end{array}\right]$$

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$$R_1 \to 2R_1 - R_2$$

$$\sim \left[\begin{array}{cc:c} 1 & 1/2 & 1 \\ 0 & 1 & 4/3 \end{array}\right] \longrightarrow \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 1 & 4/3 \end{bmatrix}$$

$$R_1 \to R_1/2$$

$$\sim \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 4/3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix}$$

$$\Rightarrow x_1 = 1/3 \quad x_2 = 4/3$$

$$\Rightarrow \text{let } x = u = \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix}$$

Next to show that $||Au - b|| \leq ||Az - b||$

$$Au = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ \frac{2}{3} + 8/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 4/3 \\ 10/3 \end{bmatrix}$$

$$\text{Now } Au - b = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \\ \frac{10}{3} \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - 1 \\ \frac{4}{3} - 2 \\ \frac{10}{3} - 3 \end{bmatrix}$$

$$Au - b = \begin{bmatrix} -1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$\|Au-b\| = \sqrt{\left(\frac{-1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3} \approx 0.8164$$

Now $\|Az-b\| = ?$ $\quad z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$Az = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$Az-b = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\|Az-b\| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3} \approx 1.732$$

$\Rightarrow \|Az-b\| \approx 1.732$

Hence

$\|Au-b\| \leq \|Az-b\|$

$\because \|Au-b\| \approx 0.8164$ and $\|Az-b\| \approx 1.732$

**Qn** Find a least-square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 0 & 7 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$$

**Soln** To find a least-square soln to $Ax = b$, we need to solve the system

$$(A^T A)x = A^T b$$

Now $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 0 & 7 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 5 & 7 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9+1 & 2+6+0 & -2+15+7 \\ 2+6+0 & 1+4+0 & -1+10+0 \\ -2+15+7 & -1+10+0 & 1+25+49 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 8 & 20 \\ 8 & 5 & 9 \\ 20 & 9 & 75 \end{bmatrix} = A^T A$$

$$A^T b = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6+6+6 \\ 3+4+0 \\ -3+10+42 \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \\ 49 \end{bmatrix}$$

Now $(A^T A)x = A^T b$

$$\begin{bmatrix} 14 & 8 & 20 \\ 8 & 5 & 9 \\ 20 & 9 & 75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \\ 49 \end{bmatrix}$$

Augmentend matrix $\quad R_1 \to R_1/14$

$$\left[\begin{array}{ccc:c} 14 & 8 & 20 & 18 \\ 8 & 5 & 9 & 7 \\ 20 & 9 & 75 & 49 \end{array}\right] \sim \begin{bmatrix} 1 & 4/7 & 10/7 & 9/7 \\ 8 & 5 & 9 & 7 \\ 20 & 9 & 75 & 49 \end{bmatrix}$$

$R_2 \to R_2 - 8R_1 \qquad R_3 \to R_3/20$

$$\begin{bmatrix} 1 & 4/7 & 10/7 & 9/7 \\ 0 & 3/7 & -17/7 & -23/7 \\ 20 & 9 & 75 & 49 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/7 & 10/7 & 9/7 \\ 0 & 3/7 & -17/7 & -23/7 \\ 1 & 9/20 & 15/4 & 49/20 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 4/7 & 10/7 & 9/7 \\ 0 & 3/7 & -17/7 & -23/7 \\ 0 & -17/140 & 65/20 & 163/140 \end{bmatrix} \sim \left[\begin{array}{ccc:c} 1 & 0 & -7 & 17/3 \\ 0 & 1 & -13 & -23/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 =$ ~~arb~~ arbitrary $\quad x_3 = c$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17/3 \\ -23/3 \\ 0 \end{bmatrix}$$

$x_1 - 7x_3 = 17/3$ $\quad x_2 + 13x_3 = -23/3$

$x_3 = c$ (let)

$x_1 = 7c + \frac{17}{3}$

$x_3 = -13c - 23/3$

Soln set $= \{(7c+17/3, -13c-23/3, c) \{ c \in \mathbb{R}\}$

$= \{(7c+\frac{17}{3}, -13c-\frac{23}{3}, c) : c \in \mathbb{R}\}$

$\Rightarrow$ Infinite numbers of least-Square soln

Q. find a least-Square Solution to the inconsistent System $Ax = b$ where

$$A = \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} \text{ and } b = \begin{bmatrix} 15 \\ 1 \\ 4 \\ 16 \end{bmatrix}$$

[Do your self]

Q. find a least-Square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 2 & 1 \end{bmatrix} \quad b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

Also verify $\|Au - b\| \le \|Az - b\|$ for $v = (3, 0)$

[Do yourself]

Q. Find a least - square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 15 \\ 1 \\ 4 \\ 16 \end{bmatrix}$$

Soln To find a least - square soln to $Ax = b$ solve the system

$$(A^T A)x = A^T b$$

$$A = \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} = \begin{bmatrix} 94 & 91 & 66 \\ 91 & 114 & 76 \\ 66 & 76 & 52 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 15 \\ 1 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 197 \\ 237 \\ 160 \end{bmatrix}$$

We get system of eqn

$$\begin{bmatrix} 94 & 91 & 66 \\ 91 & 114 & 76 \\ 66 & 76 & 52 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 197 \\ 237 \\ 160 \end{bmatrix}$$

~~Aug~~ Augmented matrix is

$R_1 \to R_1/94$

$$\left[\begin{array}{ccc:c} 94 & 91 & 66 & 197 \\ 91 & 114 & 76 & 237 \\ 66 & 76 & 52 & 160 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 91 & 114 & 76 & 237 \\ 66 & 76 & 52 & 160 \end{array}\right]$$

$R_2 \to R_2 - 91R_1$

$R_3 \to R_3 - 66R_1$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 2435/94 & 569/47 & 4351/94 \\ 0 & 569/47 & 266/47 & 1019/47 \end{array}\right]$$

$R_2 \to \frac{94}{2435} R_2 \quad R_3 \to \frac{47}{569} R_3$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 1138/2435 & 4351/2435 \\ 0 & 1 & 266/569 & 1019/569 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 1138/2435 & 4351/2435 \\ 0 & 0 & 4/2435 & 118/2435 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 1138/2435 & 4351/2435 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$R_2 \to R_2 - \frac{1138}{2435} R_3$

$$\left[\begin{array}{ccc|c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$R_1 \to R_1 - \frac{66}{94} R_3$ , $R_1 \to R_1 - \frac{91}{94} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -12 \\ 59/2 \end{bmatrix}$$

$x_1 = -7 \quad x_1 = -12 \quad x_3 = 59/2$

Q Find a least-square soln to the incon system $Ax = b$ where

$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$

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[DU CSE-2 2018]

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**Soln** To find the least squares soln to $Ax = b$ where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$$

To solve it $(A^T A)x = A^T b$

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 50 & 25 \\ 25 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix} = \begin{bmatrix} 121 \\ 51 \end{bmatrix}$$

Solve it ~~$Ax$~~ $(A^T A)x = A^T b$

$$\begin{bmatrix} 50 & 25 \\ 25 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 121 \\ 51 \end{bmatrix}$$

Augmented matrix $\quad R_1 \to R_1 / 50$

$$\left[\begin{array}{cc:c} 50 & 25 & 121 \\ 25 & 14 & 51 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 25 & 14 & 51 \end{array}\right]$$

$$R_2 \to R_2 - 25R_1 \quad \left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 0 & 25/2 & -19/2 \end{array}\right]$$

$R_2 \to R_2 \times \frac{2}{25}$ $\quad$ $R_1 \to R_1 - R_2/2$

$$\begin{bmatrix} 1 & 1/2 & \vdots & 121/50 \\ 0 & 1 & \vdots & -19/25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 14/5 \\ 0 & 1 & \vdots & -19/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14/5 \\ -19/2 \end{bmatrix} \Rightarrow x_1 = 14/5$$

$x_2 = -19/2$

$R_1 \to R_1/50$ $\quad$ $R_2 \to R_2 - 25R_1$

$$\begin{bmatrix} 1 & 1/2 & \vdots & 121/50 \\ 25 & 14 & \vdots & 51 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & \vdots & 121/50 \\ 0 & 3/2 & \vdots & -19/2 \end{bmatrix}$$

$R_2 \to \frac{2}{3} R_2$ $\quad$ $R_1 \to R_1 - R_2/2$

$$\begin{bmatrix} 1 & 1/2 & \vdots & 121/50 \\ 0 & 1 & \vdots & -19/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \vdots & 419/75 \\ 0 & 1 & \vdots & -19/3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 419/75 \\ -19/3 \end{bmatrix}$$

$\Rightarrow x_4 = 419/75 \quad x_2 = -19/3$

Qn 5 $\quad A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ and $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Find a vector $x$ satisfying inequality

~~that~~ $\|Ax - b\| \le \|Az - b\|$ Teacher's Signature

[DU CSE-7, 2019]

Sol$^n$ for find the sol$^n$ of $Ax = b$ we need to find $(A^TA)x = A^Tb$

$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ & $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

$\Rightarrow (A^TA)x = A^Tb$

$$\begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

Augmented matrix

$R_1 \to R_1/21$

$$\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 9 & 11 & 19 \end{array}\right]$$

$R_2 \to R_2 - 9R_1$ $\qquad$ $R_2 \to \frac{7}{50} R_2$

$$\left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 0 & 50/7 & 55/7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 0 & 1 & 11/10 \end{array}\right]$$

$$R_1 \to R_1 - \frac{9}{21} R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix}$$

$$\Rightarrow x_1 = 23/30 \qquad x_2 = 11/10$$

Now to show $\|Ax-b\| \leq \|Az-b\|$

$$z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Ax-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 59/30 \\ -59/30 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$Ax-b = \begin{bmatrix} -1/3 \\ -61/30 \\ -209/30 \end{bmatrix} \Rightarrow \|Ax-b\| = \sqrt{\frac{1}{9} + \left(\frac{61}{30}\right)^2 + \left(\frac{209}{30}\right)^2}$$

$$\Rightarrow \|Ax-b\| = \sqrt{\frac{1}{9} + \frac{3721}{900} + \frac{43681}{900}}$$

$$\|Ax-b\| = 7.2469$$

$$Az-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$$

$$\|Az-b\| = \sqrt{4}$$

To show $||Ax-b|| \leq ||Az-b||$ $\quad z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Ax-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 25/6 \\ 29/6 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/6 \\ -1/6 \end{bmatrix}$$

$$||Ax-b|| = \sqrt{\left(-\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{6}\right)^2 + \left(-\tfrac{1}{6}\right)^2} = \sqrt{\tfrac{1}{9} + \tfrac{1}{36} + \tfrac{1}{36}}$$

$$||Ax-b|| = \frac{\sqrt{6}}{6} \approx 0.4082$$

$$||Az-b|| = \sqrt{4+1+36} = \sqrt{41}$$

$\Rightarrow ||Az-b|| = 6.4031$

$\Rightarrow ||Ax-b|| \leq ||Az-b||$

# Orthogonality

→ two vector $x$ & $y$ in $\mathbb{R}^n$ are orthogonal if and only if $x \cdot y = 0$

Now we generalize this concept to larger sets as well.

**Defn** Orthogonal vectors

A set $S = \{v_1, v_2, v_3 \ldots v_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthogonal set of vector iff each pair of distinct vectors in this set is orthogonal

ie if and only if

$$v_i \cdot v_j = 0 \quad \forall \; i \neq j \quad 1 \leq i, j \leq k$$

**Ex** in $\mathbb{R}^3$ $S = \{(1,0,-1), (1,\sqrt{2},1), (1,-\sqrt{2},1)\}$
Is an orthogonal set?

**Soln** $S = \{\underbrace{(1,0,-1)}_{v_1}, \underbrace{(1,\sqrt{2},1)}_{v_2}, \underbrace{(1,-\sqrt{2},1)}_{v_3}\}$

$$v_1 \cdot v_2 = (1,0,-1) \cdot (1,\sqrt{2},1)$$
$$= 1 \times 1 + 0 \cdot \sqrt{2} + (-1)(1)$$
$$= 1 - 1 = 0$$

$\Rightarrow v_1 \cdot v_2 = 0 \Rightarrow v_1 \perp v_2$

$v_1 \cdot v_3 = (1, 0, -1) \cdot (1, -\sqrt{2}, 1)$
$= 1 \times 1 + 0 \times -\sqrt{2} + (-1)(1) = 0$
$\Rightarrow v_1 \cdot v_3 = 0 \Rightarrow v_1 \perp v_3$

Now $v_2 \cdot v_3 = (1, \sqrt{2}, 1) \cdot (1, -\sqrt{2}, 1)$
$= 1 \times 1 + \sqrt{2} \times -\sqrt{2} + 1 \times 1$
$= 1 - 2 + 1 = 0$

$\Rightarrow v_2 \cdot v_3 = 0 \Rightarrow v_2 \perp v_3$

$\Rightarrow$ set S is orthogonal.

Defn Orthonormal vectors (set)

A set $S = \{v_1 v_2 \cdots v_k\}$ of K distinct vectors of $\mathbb{R}^n$ is said to be an orthonormal set of vectors if and only if it is an orthogonal set and all its vectors are unit vectors i.e
$||v_i|| = 1 \quad \forall i$

Qn $S = \{\underset{v_1}{(1, 0, -1)}, \underset{v_2}{(1, \sqrt{2}, 1)}, \underset{v_3}{(1, -\sqrt{2}, 1)}\}$
S is orthonormal set ?

Soln firstly check S is orthogonal set
for this $v_1 \cdot v_2 = (1, 0, -1) \cdot (1, \sqrt{2}, 1)$

$v_1 \cdot v_2 = (1,0,-1)\cdot(1,-\sqrt{2},1) = 0$

$v_2 \cdot v_3 = \cancel{(1,\sqrt{2}} \; (1,\sqrt{2},1)\,(1,-\sqrt{2},1)$

$= 1\times 1 + \sqrt{2}\times -\sqrt{2} + 1\times 1 = 1-2+1 = 0$

$\Rightarrow v_2 \cdot v_3 = 0$

this show S is orthogonal set

Now for orthonormal set

$||v_1|| = \sqrt{1+0+1} = \sqrt{2} \neq 1$

$||v_2|| = \sqrt{1+2+1} = 2 \neq 1$

$||v_3|| = \sqrt{1+2+1} = 2 \neq 1$

$\because$ ~~$||v_i|| \neq 1$~~ $\because ||v_i|| \neq 1 \;\forall\; i = 1,2,3$

$\Rightarrow$ S is not orthonormal set

# if $||v_i|| \neq 1$ for any $i \Rightarrow$ set which contain vector $v_i$ is not ~~orthogon~~ orthonormal set.

(ii) $S = \{(2,-3),(6,4)\}$ is orthonormal set

( Do your self )

Thm orthogonality implies linear independency
Let $S = \{v_1, v_2 \ldots v_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$ then S is linearly independent set.

Proof $S = \{v_1, v_2 \ldots v_k\}$
let $\exists$ scalars $a_1, a_2 \ldots a_k \in \mathbb{R}$ s.t.

$$a_1 v_1 + a_2 v_2 \ldots + a_k v_k = 0$$

To show S is L.I
For this we show $a_1 = a_2 \ldots = a_k = 0$

$\because$ $a_1 v_1 + a_2 v_2 \ldots + a_k v_k = 0$ — (1)

~~postmultiply~~ post multiply by $v_i$ in (1)

$$(a_1 v_1 + a_2 v_2 \ldots + a_k v_k) \cdot v_i = 0 \cdot v_i = 0$$
$[1 \le i \le k]$

$$a_1 v_1 \cdot v_i + a_2 v_2 \cdot v_i \ldots + a_k v_k \cdot v_i = 0$$

$\Rightarrow$ $a_1 (v_1 \cdot v_i) + a_2 (v_2 \cdot v_i) \ldots + a_k (v_k \cdot v_i) = 0$
if we fix $i = k$

$\Rightarrow a_1 (v_1 \cdot v_k) + a_2 (v_2 \cdot v_k) \ldots + a_k (v_k \cdot v_k) = 0$
$(\because S = \{v_1, v_2 \ldots v_k\}$ is orthogonal
$\Rightarrow v_i \cdot v_j = 0 \ \forall (i \ne j))$

$\Rightarrow a_1 \cdot 0 + a_2 \cdot 0 \ldots + a_k(v_k \cdot v_k) = 0$

$\Rightarrow a_k \|v_k\|^2 = 0$ [$\because v_i \cdot v_i = \|v_i\|^2$]

$= 0$

$\Rightarrow a_k = 0$

Similarly we can show $a_1 = a_2 \ldots a_{k-1} = 0$

$\Rightarrow a_1 = a_2 \ldots = a_k = 0$

Hence set $S = \{v_1 v_2 \ldots v_k\}$ is L.I

$\Rightarrow$ If Any set is say $T = \{v_1 v_2 \ldots v_k\}$ is not L.I $\Rightarrow$ set is not orthogonal

Defn A basis for a subspace W of $\mathbb{R}^n$ is said to be an orthogonal basis for W if and only if B is an orthogonal set

Defn A basis B for W is an orthonormal basis for W if and only if B is an orthonormal set

Th. If B is an orthogonal set of n non-zero vectors in $\mathbb{R}^n$ then B an orthogonal basis for $\mathbb{R}^n$

# If B is an orthonormal set of n vectors in $\mathbb{R}^n$ then B is an orthonormal basis for $\mathbb{R}^n$

**Thm** Let $B = \{v_1, v_2 \ldots v_n\}$ is a non-empty ordered orthogonal basis for $\mathbb{R}^n$ and $v$ is any vector in $\mathbb{R}^n$ then

$$v = \frac{v \cdot v_1}{\|v_1\|^2} v_1 + \frac{v \cdot v_2}{\|v_2\|^2} v_2 + \cdots + \frac{v \cdot v_n}{\|v_n\|^2} v_n$$

In other word the coordinatization of $v$ w.r.t B is

$$[v]_B = \begin{bmatrix} \frac{v \cdot v_1}{\|v_1\|^2} \\ \frac{v \cdot v_2}{\|v_2\|^2} \\ \vdots \\ \frac{v \cdot v_n}{\|v_n\|^2} \end{bmatrix}$$

**proof** Since $B = \{v_1, v_2 \ldots v_n\}$ is an ordered orthogonal basis for $\mathbb{R}^n$ and $v \in \mathbb{R}^n$ then $\exists$ unique real numbers $a_1, a_2 \ldots a_n$ s.t

$$v = a_1 v_1 + a_2 v_2 \cdots + a_n v_n \quad \text{--- (1)}$$

For showing $[v]_B = \begin{bmatrix} \frac{v \cdot v_1}{\|v_1\|^2} \\ \vdots \\ \frac{v \cdot v_n}{\|v_n\|^2} \end{bmatrix}$

To show $a_i = \frac{v \cdot v_i}{\|v_i\|^2} \ \forall i \quad 1 \le i \le n$

$\because v = a_1 v_1 + a_2 v_2 \ldots + a_n v_n$ from ①

multiplying by $v_i$ both in ① where $1 \le i \le n$

$v \cdot v_i = (a_1 v_1 + a_2 v_2 \ldots + a_n v_n) v_i$

$= a_1 v_1 v_i + a_2 v_2 v_i \ldots + a_n v_n v_i$

$= a_1 v_1 v_i + a_2 v_2 v_i \ldots + a_i v_i v_i + a_{i+1} v_{i+1} v_i \ldots + a_n v_n v_i$

$= a_1 (v_1 v_i) + a_2 (v_2 v_i) \ldots + a_i (v_i v_i) \ldots + a_n (v_n v_i)$

$\because$ let $B = \{v_1, v_2 \ldots v_n\}$ is an orthogonal $\Rightarrow v_j \cdot v_i = 0 \ \forall j \ne i$

$= a_1 \cdot 0 + a_2 \cdot 0 \ldots + a_i v_i v_i \ldots + a_n \cdot 0$

$\Rightarrow v \cdot v_i = a_i \|v_i\|^2 \quad (\because v_i \cdot v_i = \|v_i\|^2)$

$\Rightarrow a_i = \frac{v \cdot v_i}{\|v_i\|^2} \quad \because \|v_i\| \ne 0$

Hence prove

Q. which of sets is orthogonal & orthonormal

(i) $\{(1, 0, -1), (-1, 4, 1), (2, 1, 2)\}$

(ii) $\{(\sqrt{3}/2, 1/2), (-1/2, \sqrt{3}/2)\}$

(iii) $\{(1/3, 2/3, 2/3), (2/3, 1/3, -2/3), (2/3, -2/3, 1/3)\}$

(Do yourself)

# Suppose that $\beta = \{v_1 v_2 \cdots v_n\}$ is an ordered orthonormal basis for $\mathbb{R}^n$ and $v$ is any vector in $\mathbb{R}^n$ then

$$v = (v\cdot v_1)\cdot v_1 + (v\cdot v_2)\cdot v_2 + \cdots + (v\cdot v_n)\cdot v_n$$

in other words, the coordinatization of $v$ with respect to $\beta$ is

$$[v]_\beta = \begin{bmatrix} v\cdot v_1 \\ v\cdot v_2 \\ \vdots \\ v\cdot v_n \end{bmatrix}$$

Qn verify that the set

$$\beta = \left\{ v_1 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), v_2 = \left(-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right), v_3 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \right\}$$

is an orthonormal basis for $\mathbb{R}^3$, then find $[v]_\beta = ?$ for the vector $u = (-1, 5, 3)$

Soln To show set $\beta$ is an orthonormal ~~basis for~~

firstly to show set $\beta$ is orthogonal

$$v_1 \cdot v_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right)$$

Date


$$= -\cancel{\frac{1}{6}} + 0 \times \frac{4}{3\sqrt{2}} + \cancel{\frac{1}{6}} = 0$$

$\Rightarrow v_1 v_2 = 0$

Now $v_1 v_3 = \left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right] \cdot \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$

$$= \cancel{\frac{2}{3\sqrt{2}}} + 0 \cdot \frac{1}{3} - \cancel{\frac{2}{3\sqrt{2}}} = 0$$

$\Rightarrow v_1 v_3 = 0$

Now $v_2 v_3 = \left[\frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right] \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right]$

$$= -\cancel{\frac{2}{9\sqrt{2}}} + \frac{4}{9\sqrt{2}} - \cancel{\frac{2}{9\sqrt{2}}} = 0$$

$\Rightarrow v_2 v_3 = 0$

$\Rightarrow$ set $\beta = \{v_1, v_2, v_3\}$ is orthogonal

Now $\|v_1\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$

$\|v_2\| = \sqrt{\left(\frac{-1}{3\sqrt{2}}\right)^2 + \left(\frac{4}{3\sqrt{2}}\right)^2 + \left(\frac{-1}{3\sqrt{2}}\right)^2}$

$$= \sqrt{\frac{1}{18} + \frac{16}{18} + \frac{1}{18}} = \sqrt{\frac{18}{18}} = \sqrt{1} = 1$$

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$\Rightarrow ||v_2|| = 1$

now $||v_3|| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{4+1+4}{9}} = \sqrt{1} =$

$\Rightarrow ||v_3|| = 1$

$\Rightarrow$ Set $\beta = \{v_1, v_2, v_3\}$ is orthonormal.

By using the coordinatization of v with respect to $\beta$ is

$$[v]_\beta = \begin{bmatrix} v \cdot v_1 \\ v \cdot v_2 \\ \vdots \\ v \cdot v_n \end{bmatrix}$$

$$\Rightarrow [v]_\beta = \begin{bmatrix} v \cdot v_1 \\ v \cdot v_2 \\ v \cdot v_3 \end{bmatrix} = \begin{bmatrix} (-1, 5, 3) \cdot \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) \\ (-1, 5, 3) \cdot \left(-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right) \\ (-1, 5, 3) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} + 0 - \frac{3}{\sqrt{2}} \\ \frac{1}{3\sqrt{2}} + \frac{20}{3\sqrt{2}} - \frac{3}{3\sqrt{2}} \\ -\frac{2}{3} + \frac{5}{3} + \frac{6}{3} \end{bmatrix} = \begin{bmatrix} -4/\sqrt{2} \\ 18/3\sqrt{2} \\ 9/3 \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ 3\sqrt{2} \\ 3 \end{bmatrix}$$

$$\Rightarrow [v]_\beta = \begin{bmatrix} -2\sqrt{2} \\ 3\sqrt{2} \\ 3 \end{bmatrix}$$

## Gram-Schmidt orthogonalization process

Let $\{w_1, w_2 \ldots w_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. The Gram-Schmidt process constructs a new set $\{v_1, v_2 \ldots v_k\}$ of vectors such that $\{v_1, v_2 \ldots v_k\}$ becomes an orthogonal basis for span$\{w_1, w_2 \ldots w_k\}$.

Steps for Gram-Schmidt Process

Step 1 Let $v_1 = w_1$

Step 2 $v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) v_1$

Step 3 $v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|^2}\right) v_2$

$\vdots$

Continue this process upto $v_k$, then the resulting set $\{v_1, v_2 \ldots v_k\}$ forms an orthogonal basis for the subspace spanned by the set $\{w_1, w_2 \ldots w_k\}$

* If $B = \{w_1, w_2 \ldots w_k\}$ be a basis for a subspace $W$ of $\mathbb{R}^n$, then the set $T = \{v_1 \ldots v_k\}$ obtained by applying the Gram-Schmidt process to $B$ is an orthogonal basis for $W$.

Qn $\beta = \{(2,1,0,-1), (1,0,2,-1), (0,-2,1,0)\}$ of $\mathbb{R}^4$ find an orthogonal bases for the subspace of $\mathbb{R}^4$ spanned by $\beta$.

Soln $\beta = \{(2,1,0,-1), (1,0,2,-1), (0,-2,1,0)\}$

By independence test method

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & -4 \\ 0 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -7 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\because$ each colm has pivot

$\Rightarrow$ set $\beta$ is L.I

Now we will apply the Gram-Schmidt process to find an orthogonal basis for $W$

Let $w_1 = (2,1,0,-1)$, $w_2 = (1,0,2,-1)$

and $w_3 = (0,-2,1,0)$

Step 1 $v_1 = w_1 = (2,1,0,-1)$

Step 2 $v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) v_1$

$= [1,0,2,-1] - \left(\frac{(1,0,2,-1)\cdot(2,1,0,-1)}{\|(2,1,0,-1)\|^2}\right)(2,1,0,-1)$

$= [1,0,2,-1] - \left[\frac{1\times 2 + 0\times 1 + 2\times 0 + 1\times 1}{(\sqrt{4+1+0+1})^2}\right](2,1,0,-1)$

$= (1,0,2,-1) - \frac{3}{6}(2,1,0,-1)$

$= (1,0,2,-1) - (1, \tfrac{1}{2}, 0, -\tfrac{1}{2})$

$= (1-1, 0-\tfrac{1}{2}, 2-0, -1+\tfrac{1}{2}) = (0, -\tfrac{1}{2}, 2, -\tfrac{1}{2})$

$\therefore v_2 = (0, -\tfrac{1}{2}, 2, -\tfrac{1}{2}) = (0,-1,4,-1)$

Now Step 3 $v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|^2}\right) v_2$

$= (0,-2,1,0) - \frac{(0,-2,1,0)\cdot(2,1,0,-1)}{\|(2,1,0,-1)\|^2} \cdot (2,1,0,-1)$

$- \frac{(0,-2,1,0)\cdot(0,-1,4,-1)}{\|(0,-1,4,-1)\|^2} \cdot (0,-1,4,-1)$

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$$v_3 = (0,-2,1,0) - \left(\frac{0+2+0-0}{6}\right)\cdot(2,1,0,-1)$$

$$- \left(\frac{0+2+4-0}{18}\right)\cdot(0,-1,4,-1)$$

$$= (0,-2,1,0) + \frac{1}{3}(2,1,0,-1) - \frac{1}{3}(0,-1,$$

$$= (0,-2,1,0) + \left(\frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}\right) + \left(0, +\frac{1}{3}, -\frac{4}{3},\right.$$

$$= (0,-2,1,0) + \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}, 0\right)$$

$$= \left(\frac{2}{3}, -\frac{4}{3}, -\frac{1}{3}, 0\right)$$

$$v_3 = \left[\frac{2}{3}, -\frac{4}{3}, -\frac{1}{3}, 0\right] = [2,-4,-1,0]$$

Hence the set

$\{v_1, v_2, v_3\} = \{(2,1,0,-1), (0,-1,4,-1), (2,-4,$

is an orthogonal bases ~~for~~ W.

Qn $v = [-2,1]$ $\beta = \left\{\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)\right\}$

find $[v]_\beta$ [Do yourself]

$v = (4,-1,2)$ and

$\beta = \left\{ \left(\frac{3}{7}, -\frac{6}{7}, -\frac{2}{7}\right), \left(\frac{2}{7}, \frac{2}{7}, -\frac{6}{7}\right), \left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right) \right\}$

find $[v]_\beta$

[Do yourself].

$v = (3,1,-2)$ $\beta = \left\{ \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \right\}$

find $[v]_\beta$

Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by the set

$\beta = \{ (2,1,0,-1), (1,1,1,1), (1,-2,1,1) \}$

[Do yourself]

Use the Gram-Schmidt process to find an orthogonal basis for the subspace of $\mathbb{R}^3$ spanned by the set

$\beta = \{ (3,1,-2), (5,-5,-1), (0,1,0) \}$

[Do yourself]

Orthogonal compliment

Def$^n$ Let $W$ be a subspace of $\mathbb{R}^n$ the orthogonal compliment of $W$, denoted by $W^\perp$ is the set of all vectors of $\mathbb{R}^n$ that are orthogonal to every vector in $W$ that is

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0 \ \forall w \in W\}$$

Theo If $W$ is a subspace of $\mathbb{R}^n$, then $v \in W^\perp$ if and only if $v$ is orthogonal to every vector in a spanning set for $W$

Proof Let $S = \{w_1, w_2 \ldots w_k\}$ be a spanning set set for $W$

i.e $span(S) = W$

Let $v \in W^\perp$ to show $v \cdot w_i = 0 \ \forall \ 1 \le i \le k$

since $v \in W^\perp \Rightarrow v \cdot w = 0 \ \forall w \in W$

$\because w$ is arbitrary in $W \Rightarrow v \cdot w_i = 0 \ \forall 1 \le i \le k$

Conversely let $v \cdot w_i = 0 \ \forall \ 1 \le i \le k$

let $w \in W = span(S)$

To show $v$ is orthogonal to each vector in spanning set for $W$

$\because w \in span(S) \Rightarrow \exists$ scalars $a_1, a_2 \ldots a_k$ such that

$$w = a_1 w_1 + a_2 w_2 \ldots + a_k w_k \quad -(1)$$

multiplying v in ①

$v \cdot w = v(a_1 w_1 + a_2 w_2 \cdots + a_k w_k)$

$= v a_1 w_1 + v a_2 w_2 \cdots + v a_k w_k$

$= a_1(v \cdot w_1) + a_2(v \cdot w_2) \cdots + a_k(v \cdot w_k)$

$= a_1 \cdot (0) + a_2 \cdot (0) \cdots + a_k \cdot (0)$

$[\because v w_i = 0 \ \forall \ 1 \le i \le k]$

$= 0$

$\therefore v \cdot w = 0 \Rightarrow v \in W^{\perp}$

Hence v is orthogonal to each vector in spaning set for w.

$\xrightarrow{}$ subspace of $\mathbb{R}^3$

Q. $W = \{(a, 0, c) : a, c \in \mathbb{R}\}$ TO show

$\dim W + \dim W^{\perp} = \dim \mathbb{R}^3$.

Ans Since $W = \{(a, 0, c) : a, c \in \mathbb{R}\}$

$= \{a(1, 0, 0) + c(0, 0, 1) : a, c \in \mathbb{R}\}$

$= \text{span}\{(1, 0, 0), (0, 0, 1)\}$

$\because$ if w is a subspace of $\mathbb{R}^n$ then $u \in W^{\perp}$ iff u is orthogonal to each vector in a spaning set for w

$\therefore$ if $(x, y, z) \in W^{\perp} \Rightarrow (x, y, z) \cdot (1, 0, 0) = 0$

$\Rightarrow x = 0$

if $(x, y, z) \in W^{\perp} \Rightarrow (x, y, z) \cdot (0, 0, 1) = 0$

$z = 0$

$\therefore W^{\perp} = \{(x,y,z) : x, y, z \in \mathbb{R}\}$
$= \{(0,y,0) : y \in \mathbb{R}\}$
$= \{y(0,1,0) : y \in \mathbb{R}\}$
$= \text{span}\{(0,1,0)\}$

$\therefore \dim W^{\perp} = 1$

Since $W = \{(a,0,c) : a, c \in \mathbb{R}\}$
$= \{a(1,0,0) + c(0,0,1) : a, c \in \mathbb{R}\}$

$= \text{span}\{(1,0,0), (0,0,1)\}$
$\because \{(1,0,0), (0,0,1)\}$ is L-I set

$\Rightarrow \dim W = 2$

Hence $\dim W + \dim W^{\perp} = 2+1 = 3 = \dim \mathbb{R}^3$
$\Rightarrow \underline{\dim W + \dim W^{\perp} = \dim \mathbb{R}^3}$

Q: Let $W = \{a(-1,2,3) : a \in \mathbb{R}\}$ is subspace of $\mathbb{R}^3$. To show $\dim W + \dim W^{\perp} = \dim \mathbb{R}^3$

Sol: Since $W = \{a(-1,2,3) : a \in \mathbb{R}\}$
$= \text{span}\{(-1,2,3)\}$

$\Rightarrow \dim W = 1$
By using theo if $W$ is subspace of $\mathbb{R}^n$ then $u \in W^{\perp}$ iff $u$ is orthogonal to each vector in a spaning set for $W$.

Let $W^{\perp} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$

By two if $(x, y, z) \in W^{\perp} \Rightarrow (x, y, z)(-1, 2, 3) = 0$

$\Rightarrow -x + 2y + 3z = 0$

$\Rightarrow x = 2y + 3z$

$\Rightarrow W^{\perp} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$

$= \{(2y + 3z, y, z) : y, z \in \mathbb{R}\}$

$= \{(2y, y, 0) + (3z, 0, z) : y, z \in \mathbb{R}\}$

$= \{y(2, 1, 0) + z(3, 0, 1) : y, z \in \mathbb{R}\}$

$=$ spanning $\{(2, 1, 0), (3, 0, 1)\}$

Now
$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & -3 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3/2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ $\because$ each colm has pivot

$\Rightarrow \{(2, 1, 0), (3, 0, 1)\}$ is L.I

$\Rightarrow \dim W^{\perp} = 2$

Hence $\dim W + \dim W^{\perp} = 1 + 2 = 3$

$= \dim \mathbb{R}^3$

$\Rightarrow \dim W + \dim W^{\perp} = \dim \mathbb{R}^3$

Thm Let $W$ be a subspace of $\mathbb{R}^n$ Then $W^\perp$ is a subspace of $\mathbb{R}^n$ and $W \cap W^\perp = \{0\}$

Proof since zero vector is orthogonal to each vector in $\mathbb{R}^n \Rightarrow 0 \in W^\perp \Rightarrow W^\perp \neq \phi$

Now to show $W^\perp$ is subspace of $\mathbb{R}^n$

let $x_1, x_2 \in W^\perp$ to show $x_1 + x_2 \in W^\perp$, $w \in W$

$(x_1+x_2).w = x_1.w + x_2.w = 0+0 = 0$

$\Rightarrow x_1 + x_2 \in W^\perp$

let $x \in W^\perp$ & $c \in \mathbb{R}$ to show $cx \in W^\perp$ $\forall w \in W$

Now $(cx).w = c(x.w) = c(0) = 0$

$\Rightarrow cx \in W^\perp$

$\Rightarrow W^\perp$ is a subspace of $\mathbb{R}^n$

Now to show $W \cap W^\perp = \{0\}$

let $w \in W \cap W^\perp \Rightarrow w \in W$ and $w \in W^\perp$

$\Rightarrow w.w = 0$ ($\because$ if $x \in W$, $y \in W^\perp \Rightarrow x.y = 0$)

$\Rightarrow \|w\|^2 = 0$

$\Rightarrow w = 0$ $\Rightarrow W \cap W^\perp \subseteq \{0\}$ — ①

$\because \{0\} \in W$, and $\{0\} \in W^\perp \Rightarrow \{0\} \in W \cap W^\perp$

$\Rightarrow \{0\} \subseteq W \cap W^\perp$ — ②

From ② & ① $W \cap W^\perp = \{0\}$ [$\because$ if $A \subseteq B$, $B \subseteq A \Rightarrow A = B$]

**Thm** Let $W$ be a subspace of $\mathbb{R}^n$. Let $\{v_1, v_2 \ldots v_k\}$ be an orthogonal basis for $W$ contained in an orthogonal basis $\{v_1, v_2 \ldots v_k, v_{k+1} \ldots v_n\}$. Then $\{v_{k+1}, v_{k+2} \ldots v_n\}$ is an orthogonal basis for $W^\perp$. [ not proof]

# Let $W$ be a subspace of $\mathbb{R}^n$. then
$\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^n)$

**Corollary** Let $W$ be a subspace of $\mathbb{R}^n$
Then $(W^\perp)^\perp = W$

Proof Since $W \subseteq (W^\perp)^\perp$

To show $W = (W^\perp)^\perp$ it is enough to show that $\dim(W) = \dim(W^\perp)^\perp$

$\because \dim W + \dim W^\perp = \dim \mathbb{R}^n = n$ —①

$\Rightarrow \dim(W^\perp) + \dim((W^\perp))^\perp = n$

$\Rightarrow \dim(W^\perp)^\perp = n - \dim W^\perp$
$= \dim W$ (from ①)

$\Rightarrow \dim(W^\perp)^\perp = \dim W$

$\Rightarrow W = (W^\perp)^\perp$

**Qn** $W = \{(x,y,z) \in \mathbb{R}^3 : 3x - y + 4z = 0\}$ is subspace of $\mathbb{R}^3$ find the orthogonal complement $W^\perp$ & To show
$\dim(W) + \dim(W^\perp) = \dim \mathbb{R}^3$
[DU GE-2-2017]

Sol: $\because W = \{(x,y,z) \in \mathbb{R}^3 : 3x - y + 4z = 0\}$

$= \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \cdot (3,-1,4) = 0\}$

$\Rightarrow$ W is the set of all vectors orthogonal to $(3,-1,4)$

$\Rightarrow$ W is orthogonal to subspace $Y = \text{span}\{(3,-1,4)\}$

$\Rightarrow W = Y^{\perp}$ (By def$^n$ of orthogonal complement)

$\Rightarrow W^{\perp} = (Y^{\perp})^{\perp} = Y = \text{span}\{(3,-1,4)\}$

$\Rightarrow W^{\perp} = \text{span}\{(3,-1,4)\}$

$\Rightarrow$ orthogonal complement is $= \text{span}\{(3,-1,4)\}$

$\Rightarrow$ basis for orthogonal complement is $\{(3,-1,4)\}$

$\dim W^{\perp} = 1$

Now $W = \{(x,y,z) \in \mathbb{R}^3 : 3x - y + 4z = 0\}$

$= \{(x,y,z) : y = 3x + 4z\}$

$= \{(x, 3x+4z, z) : x, z \in \mathbb{R}\}$

$= \{(x, 3x, 0) + (0, 4z, z) : x, z \in \mathbb{R}\}$

$= \{x(1,3,0) + z(0,4,1) : x, z \in \mathbb{R}\}$

$= \text{span}\{(1,3,0), (0,4,1)\}$

$\because \{(1,3,0), (0,4,1)\}$ is L.I. set

$\Rightarrow \dim W = 2$

Hence $\dim W + \dim W^{\perp} = 2 + 1 = 3 = \dim \mathbb{R}^3$

$\Rightarrow \dim W + \dim W^{\perp} = \dim \mathbb{R}^3$

Qn $W = \{(x,y,z) : 2x - 5y + z = 0\}$ is subspace of $\mathbb{R}^3$ find a basis for the orthogonal complement $W^\perp$ and verify that
$\dim W + \dim W^\perp = \dim \mathbb{R}^3$ [Do yourself]
[D.U WBF-2 2016, 2018, 2019]

Qn $W = \text{span}\{(-1,5,2)\}$ find the basis for $W^\perp$ and verify that
$\dim W + \dim W^\perp = \dim \mathbb{R}^2$
[Do your self]

Qn $W = \{(x,y,z) : -x + 4y - 2z = 0\}$ is subspace of $\mathbb{R}^3$ find the bases for $W^\perp$ and To show $\dim W + \dim W^\perp = \dim \mathbb{R}^3$
[Do your self]

Qn $W = \{(x,y,z) : x - y + 3z = 0\}$ is subsp of $\mathbb{R}^3$ find the basis for $W^\perp$ and To shows $\dim W + \dim W^\perp = \dim \mathbb{R}^3$
[Do your self]

Qn $W = \text{span}\{(-1,3,2)\}$ find bases for $W^\perp$ & To show $\dim W + \dim W^\perp = 3$

Q. Find a least-square soln to the inconsistent system $Ax = b$ where

$$A = \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 15 \\ 1 \\ 4 \\ 16 \end{bmatrix}$$

Soln To find a least-square soln to $Ax = b$ solve the system

$$(A^T A)x = A^T b$$

$$A = \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} \quad A^T = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 7 & 7 & 5 \\ 4 & 0 & 1 \\ 2 & 1 & 1 \\ 5 & 8 & 5 \end{bmatrix} = \begin{bmatrix} 94 & 91 & 66 \\ 91 & 114 & 76 \\ 66 & 76 & 52 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 7 & 4 & 2 & 5 \\ 7 & 0 & 1 & 8 \\ 5 & 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 15 \\ 1 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 197 \\ 237 \\ 160 \end{bmatrix}$$

We get system of eqn

$$\begin{bmatrix} 94 & 91 & 66 \\ 91 & 114 & 76 \\ 66 & 76 & 52 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 197 \\ 237 \\ 160 \end{bmatrix}$$

~~Aug~~ Augmented matrix is

$R_1 \to R_1/94$

$$\left[\begin{array}{ccc:c} 94 & 91 & 66 & 197 \\ 91 & 114 & 76 & 237 \\ 66 & 76 & 52 & 160 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 91 & 114 & 76 & 237 \\ 66 & 76 & 52 & 160 \end{array}\right]$$

$R_2 \to R_2 \to 91R_1$

$R_3 \to R_3 - 66R_1$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 2435/94 & 569/47 & 4351/94 \\ 0 & 569/47 & 266/47 & 1019/47 \end{array}\right]$$

$R_2 \to \frac{94}{2435} R_2 \quad R_3 \to \frac{47}{569} R_3$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 192/94 \\ 0 & 1 & 1130/2435 & 4351/2435 \\ 0 & 1 & 266/569 & 1019/569 \end{array}\right]$$

$R_3 \to R_3 \to R_2$

$$\left[\begin{array}{ccc:c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 1130/2435 & 4351/2435 \\ 0 & 0 & 4/2435 & 110/2435 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 1138/2435 & 4351/2435 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$R_2 \to R_2 - \frac{1138}{2435} R_3$

$$\left[\begin{array}{ccc|c} 1 & 91/94 & 66/94 & 197/94 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$R_1 \to R_1 - \frac{66}{94} R_3$ , $R_1 \to R_1 - \frac{91}{94} R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 59/2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -12 \\ 59/2 \end{bmatrix}$$

$x_1 = -7 \quad x_1 = -12 \quad x_3 = 59/2$

Q. Find a least-square soln to the incon system $Ax = b$ where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$$



[DU CSE-2 2018]

3.17

**Soln** To find the least squares soln to $Ax = b$ where

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix}$$

To solve it $(A^T A)x = A^T b$

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 3 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 50 & 25 \\ 25 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 5 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \\ 4 \end{bmatrix} = \begin{bmatrix} 121 \\ 51 \end{bmatrix}$$

solve it ~~Ax~~ $(A^T A)x = A^T b$

$$\begin{bmatrix} 50 & 25 \\ 25 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 121 \\ 51 \end{bmatrix}$$

Augmented matrix $\quad R_1 \to R_1/50$

$$\left[\begin{array}{cc:c} 50 & 25 & 121 \\ 25 & 14 & 51 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 25 & 14 & 51 \end{array}\right]$$

$$R_2 \to R_2 - 25R_1 \quad \left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 0 & 25/2 & -19/2 \end{array}\right]$$

$R_2 \to R_2 \times \frac{2}{25}$ $\qquad$ $R_1 \to R_1 - R_2/2$

$$\left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 0 & 1 & -19/25 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 14/5 \\ 0 & 1 & -19/2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14/5 \\ -19/2 \end{bmatrix} \Rightarrow x_1 = 14/5$$

$$x_2 = -19/2$$

$R_1 \to R_1/50$ $\qquad$ $R_2 \to R_2 - 25R_1$

$$\left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 25 & 14 & 51 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 0 & 3/2 & -19/2 \end{array}\right]$$

$R_2 \to \frac{2}{3} R_2$ $\qquad$ $R_1 \to R_1 - R_2/2$

$$\left[\begin{array}{cc:c} 1 & 1/2 & 121/50 \\ 0 & 1 & -19/3 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 419/75 \\ 0 & 1 & -19/3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 419/75 \\ -19/3 \end{bmatrix}$$

$$\Rightarrow x_1 = 419/75 \qquad x_2 = -19/3$$

Qn 5 $\quad A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ and $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Find a vector $x$ satisfying inequality

~~that~~ $\|Ax - b\| \le \|Az - b\|$ $\quad$ Teacher's Signature ____

[DU CSE-7, 2019]

Sol^n for find the sol^n of $Ax = b$ we need to find $(A^T A)x = A^T b$

$A = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix}$ & $b = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

$\Rightarrow (A^T A)x = A^T b$

$$\begin{bmatrix} 21 & 9 \\ 9 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

Augmented matrix

$R_1 \to R_1 / 21$

$$\left[\begin{array}{cc|c} 21 & 9 & 26 \\ 9 & 11 & 19 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 9 & 11 & 19 \end{array}\right]$$

$R_2 \to R_2 - 9R_1$ $R_2 \to \frac{7}{50} R_2$

$$\left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 0 & 50/7 & 55/7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 9/21 & 26/21 \\ 0 & 1 & 11/10 \end{array}\right]$$

$R_1 \rightarrow R_1 - \frac{9}{21} R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 23/30 \\ 0 & 1 & 11/10 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix}$$

$\Rightarrow x_1 = 23/30 \quad x_2 = 11/10$

Now to show $\|Ax-b\| \le \|Az-b\|$

$$z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Ax-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 59/30 \\ -59/30 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$Ax-b = \begin{bmatrix} -1/3 \\ -61/30 \\ -209/30 \end{bmatrix} \Rightarrow \|Ax-b\| = \sqrt{\frac{1}{9} + \left(\frac{61}{30}\right)^2 + \left(\frac{209}{30}\right)^2}$$

$$\Rightarrow \|Ax-b\| = \sqrt{\frac{1}{9} + \frac{3721}{900} + \frac{43681}{900}}$$

$$\|Ax-b\| = 7.2469$$

$$Az-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$$

$\|Az-b\| = \sqrt{4}$

To show $||Ax-b|| \leq ||Az-b||$ $\quad z = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Ax-b = \begin{bmatrix} 1 & -1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 23/30 \\ 11/10 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 \\ 25/6 \\ 29/6 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1/6 \\ -1/6 \end{bmatrix}$$

$$||Ax-b|| = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(-\frac{1}{6}\right)^2} = \sqrt{\frac{1}{9} + \frac{1}{36} + \frac{1}{36}}$$

$$||Ax-b|| = \frac{\sqrt{6}}{6} \approx 0.4082$$

$$||Az-b|| = \sqrt{4+1+36} = \sqrt{41}$$

$\Rightarrow ||Az-b|| = 6.4031$

$\Rightarrow ||Ax-b|| \leq ||Az-b||$